NUMERICAL SIMULATION OF A TWO-DIMENSIONAL SPATIALLY EVOLVING TURBULENT CHANNEL FLOW WITH THE USE OF THE LAGRANGIAN SAMPLE MEAN METHOD

Ricardo Carvalho de Almeida
Universidade Federal do Paraná
Departamento de Engenharia Mecânica
Caixa Postal 19040
Curitiba – PR - Brasil
CEP 81531-980
e-mail: rcalmeida@terra.com.br

José Luis Drummond Alves
Universidade Federal do Rio de Janeiro
Instituto Alberto Luiz Coimbra de Pós-Graduação e Pesquisa de Engenharia
Programa de Engenharia Civil
Caixa Postal 68552
Rio de Janeiro - RJ – Brasil
CEP: 21949-900
e-mail: jalves@lamce.ufrj.br

Abstract. *In this work the Lagrangian Sample Mean method (LSM) is employed to simulate a spatially evolving turbulent channel flow. The LSM method is based on the hypothesis that the small scale motions in a fluid flow are responsible for the homogenization of the physical properties in infinitesimal fluid parcels. Such a process is mathematically represented by a linear averaging operator. When the LSM is applied to the Navier-Stokes equations for an incompressible flow, in the Lagrangian frame of reference, it is possible to reduce the number of degrees of freedom to be represented in the flow system, while keeping the number of unknowns the same as the number of available equations, avoiding therefore the closure problem. Results of numerical simulation of a plane Poiseuille flow at* $Re_\tau = 237$ *with the use of the LSM method are presented. The results obtained show that the simulation was able to reproduce important large scale features of a plane channel flow, such as laminar-turbulent spatial transition, the "law of the wall" normalized velocity profile, and significant inertial subranges in energy spectra. Those results indicate that the LSM method may be competitive with other methods presently applied to simulate viscous fluid flows.*

Keywords*: Lagrangian Sample Mean, computational fluid dynamics, channel flow, semi-Lagrangian method, boundary-layer flows.*

1. INTRODUCTION

This paper presents a new approach to treat the N-S equations for numerical simulation of incompressible viscous fluid flows, which is applicable both to laminar and turbulent conditions. The method is named *Lagrangian Sample Mean* (LSM) and it is based on the physical interpretation of the small scale motions in fluid flows as being responsible for the homogenization of physical properties in the smaller scales. The method is intended to be of general application since it does not depend on the use of closure schemes, making empirical coefficients unnecessary. In this paper the problem will be treated in two dimensions only and it is not expected that all the features of real flows will be exactly reproduced. However, it is anticipated that the main large scale characteristics of the flows will be suitably reproduced by the numerical experiments.

Section 2 will introduce the concept of the LSM, its definition and the application to the N-S equations. The numerical implementation of the LSM method is described in Section 3. The method will be applied to simulate a spatially evolving turbulent channel flow. The results of the simulations will be discussed in Section 4 and the conclusions of the work will be presented in Section 5.

2. THE LAGRANGIAN SAMPLE MEAN

2.1. Conceptual Formulation

According to the turbulence theory for the atmospheric boundary layer developed by Stull (1988), the larger eddies are responsible for the transport of fluid along finite distances whereas the smaller eddies mix fluid properties in small scales. Starting from this idea the LSM method assumes that the mixing in small scales causes homogenization of the

fluid properties in those scales. Still focusing on small scales, one can suppose that the fluid particles are continuously interacting and exchanging properties among them. Then, it can be considered that infinitesimal fluid parcels will have their physical properties homogenized after each instantaneous interaction. The outcome of this homogenization process is that the physical properties in an infinitesimal fluid parcel will be averaged out, with the resulting averages being determined from the values the properties had in each fluid particle before the interactions. The simplest way to represent an average as a linear operator is through the arithmetic mean.

In incompressible Newtonian fluid flows the advection terms in the momentum equations are non-linear. Those terms appear in the equations when they are written in the Eulerian frame of reference. On the other hand, when the equations are written in the Lagrangian frame of reference the non-linearities are not explicit, because they are (implicitly) included in the total derivative.

In summary, the LSM is based on the hypothesis that it is possible to obtain the aggregation of the system that represents a fluid flow in the Lagrangian frame of reference, by describing the homogenization of physical properties in infinitesimal fluid parcels with the use of a linear operator.

2.2. Mathematical Formulation

Consider an infinitesimal fluid parcel which is made up of a finite number *n* of constituent particles that move with the local fluid velocity. The parcel therefore defines a material volume limited by a material surface, with no mass flux across it (Pope, 2000). The fluid parcel is sufficiently small so that it is possible to define a point in it, located at position $\{x\}$, which is statistically representative of the fluid parcel. This point will assume the mean value of the properties of the surrounding particles enclosed within the fluid parcel (Aris, 1962). Consider also that at time *t* we obtain a sample without replacement of the values of a certain property *q* from *N* particles that occupy positions $\{r_i\}$ in the fluid parcel, with $n >> N$. Each value obtained will be $q_i = q(\{r_i\},t)$. Under these conditions each value q_i can be considered to be an independent random variable with the same probability distribution. Therefore the sample of values will be a random sample and it will describe a collection of observations taken from the same population. The sample mean $\langle q \rangle(\{x\}, t)$ of the values q_i is the minimum variance unbiased estimator of the population average of property *q*, in the fluid parcel at position $\{x\}$, at time *t* (Devore, 1987). It is defined as

$$\langle q \rangle(\{x\},t) \equiv \frac{\sum_{i=1}^{N} q_i}{N} \tag{1}$$

Since the sample mean was obtained from fluid particles that move with the local fluid velocity i.e., in the Lagrangian (or material) frame of reference, $\langle q \rangle(\{x\},t)$ will be named *Lagrangian Sample Mean* (LSM). The physical interpretation of the LSM is that it represents the homogenization of properties of a fluid parcel due to interaction (mixing) of its several constituent particles. After the interactions, a certain physical property in the fluid parcel will assume a mean value computed from the value the property had in each of the individual particles before the interactions.

Using the definition in Eq. (1) the following properties can be easily proven:

I) Conservation of a constant *a*:

$$\langle a \rangle = a \tag{2}$$

II) Linearity: let $\psi_i = \psi(\{r_i\},t)$ and $\phi_i = \phi(\{r_i\},t)$

$$\langle \psi + \phi \rangle = \langle \psi \rangle + \langle \phi \rangle \tag{3}$$

$$\langle a\psi \rangle = a\langle \psi \rangle \tag{4}$$

III) Commutation with partial derivative: let $s = x_i$

$$\left\langle \frac{\partial q}{\partial s} \right\rangle = \frac{\partial \langle q \rangle}{\partial s} \tag{5}$$

$$\left\langle \frac{\partial^2 q}{\partial s^2} \right\rangle = \frac{\partial^2 \langle q \rangle}{\partial s^2} \tag{6}$$

IV) Commutation with total derivative: let $\frac{Dq}{Dt} = \lim_{\delta t \to 0} \frac{q(t+\delta t) - q(t)}{\delta t}$

$$\left\langle \frac{Dq}{Dt} \right\rangle = \frac{1}{N} \sum_{1=1}^{N} \lim_{\delta t \to 0} \frac{q_i(t+\delta t) - q_i(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\langle q \rangle (t+\delta t) - \langle q \rangle (t)}{\delta t} = \frac{D\langle q \rangle}{Dt} \tag{7}$$

Property IV is of paramount importance since it states that when we remain in the Lagrangian frame of reference the LSM aggregates the system without producing new unknowns. This is possible because the LSM operator remains linear in the Lagrangian frame of reference.

2.3. Application of the LSM to the N-S Equations

Applying the LSM operator in Eq. (1), and properties I through IV to the N-S equations, we obtain

$$\frac{\partial \langle u_i \rangle}{\partial x_i} = 0; \quad i = 1,2,3 \tag{8}$$

$$\frac{D\langle u_i \rangle}{Dt} + \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_i} - \nu \nabla^2 \langle u_i \rangle = 0; \quad i = 1,2,3 \tag{9}$$

Equations (8) and (9) will be called the *aggregated N-S equations*, where u_i are the *x, y* and *z* velocity components, *p* is the pressure and *v* is the kinematic viscosity of the fluid. It can be seen that they have the same functional form of the original N-S equations. Note that in the conceptualization of the LSM no distinction was made between laminar and turbulent flows. Besides, there was no analogy to molecular interactions as occurs in the eddy viscosity hypothesis. In the case the assumptions adopted to obtain the aggregated N-S equations are correct, it will be possible to predict the values of the aggregated variables directly, with a reduced number of degrees of freedom to be represented in the flow system. Since the number of unknowns is equal to the number of equations available there is no closure problem. Therefore, it will not be necessary to use approximations based on models and empirical coefficients.

3. NUMERICAL IMPLEMENTATION OF THE LSM METHOD

One of the fundamental assumptions of the LSM method is that the flow must be represented in the Lagrangian frame of reference. Consequently, the numerical solution of the N-S aggregated equations requires the use of that frame of reference for consistency. To abide by this requirement the numerical implementation of the LSM will employ the semi-Lagrangian method.

Another critical aspect in the implementation of the LSM method is to define how to obtain the sample mean. According to Wilcox (2000) the values of the properties of the flow at grid points in a numerical simulation represent average values. He analyzed the approximation by centered finite difference for the first derivative of a continuous variable $q(x)$, in a grid with spacing *h* between the grid points, and concluded that

$$\frac{q(x+h) - q(x-h)}{2h} = \frac{d}{dx}\left[\frac{1}{2h} \int_{x-h}^{x+h} q(\xi) d\xi \right] \tag{10}$$

The expression above states that the centered finite difference approximation can be interpreted as an operator that produces the derivative of the mean value of $q(x)$. Taking advantage of this property, it will not be necessary to actually calculate the sample mean by using Eq. (1). Instead, the spatial derivatives will be approximated by finite differences, and the averaging operation implicit in this approximation will play the role of the LSM operator [Eq. (1)] yielding the mean value of the variables in discrete fluid parcels represented by the grid cells. These mean values will be the LSM's and the gridpoints will be assumed to be the positions $\{x\}$ representative of the fluid parcels after the particle interactions.

The semi-Lagrangian three-time-level scheme associated to the semi-implicit algorithm proposed by Robert *et al.* (1985) will be used in this work. As mentioned before, the flow problem will be restricted to two dimensions.

4. SIMULATION OF A TWO-DIMENSIONAL TURBULENT CHANNEL FLOW

In the experiment simulating a turbulent channel flow with the use of the LSM method the aggregated N-S equations (8) and (9) will be used in component (*x*-*y*) form. The computational domain simulates a two-dimensional channel with

length L = 5 m, and uniform height 2δ = 0.025 m, with the inflow boundary on the left end of the channel. The grid is uniform along the x and y directions, with $\Delta y << \Delta x$.

Along the inflow boundary the u component was defined according to the following expression for the profile of the stream-wise velocity component (u) in a fully developed laminar channel flow (Kundu, 1990):

$$u = \frac{y}{\mu}\frac{dp}{dx}\left(\delta - \frac{y}{2}\right) \tag{11}$$

where μ is the dynamic viscosity of the fluid and dp/dx is the longitudinal pressure gradient. The v velocity component at the inflow boundary was set equal to zero in order to define the flow as parallel to the walls at the channel entrance. These inflow boundary conditions are kept constant along the simulation. At the channel exit the u and v components are predicted at each time step, using the fact that the flow is continuously outwards along that boundary. At the upper and lower boundaries (walls) the no-slip (u = 0) and impenetrability (v = 0) boundary conditions were used. The pressure p was prescribed along the walls as linearly decreasing along the x direction. In the semi-Lagrangian method, spatial bi-cubic interpolations were used. Centered finite differences were used for spatial discretization of the equations.

It is important to point out that differently from the usual procedure employed when simulating channel flows with the use of Direct Numerical Simulation (DNS) or Large Eddy Simulation (LES), in the experiment presented in this study the boundary conditions are physical, instead of periodic. Therefore, it is expected that the simulation will replicate the spatial evolution of the flow along the x direction.

The longitudinal velocity component at the channel centerline U_c ($y = \delta$) = 5.25 m/s, and the flow characteristic Reynolds number Re_τ = 237, refer to the position $x = L/2$ = 2.5 m. The simulated fluid is air at 293 K. It is important to point out that neither parameterizations nor turbulent models were used in the simulation.

The simulated flow time was 67.86 s. Temporal means of the variables (represented by overbars) were calculated for the interval between 9.69 s and 67.86 s.

Figure 1 shows the time evolutions of the domain averaged enstrophy and kinetic energy (per unit mass), respectively. After an initial adjustment phase, the flow reaches a quasi-stationary state, after $t \approx$ 7s. It can be also seen that there was no indication of numerical instability during the simulation.

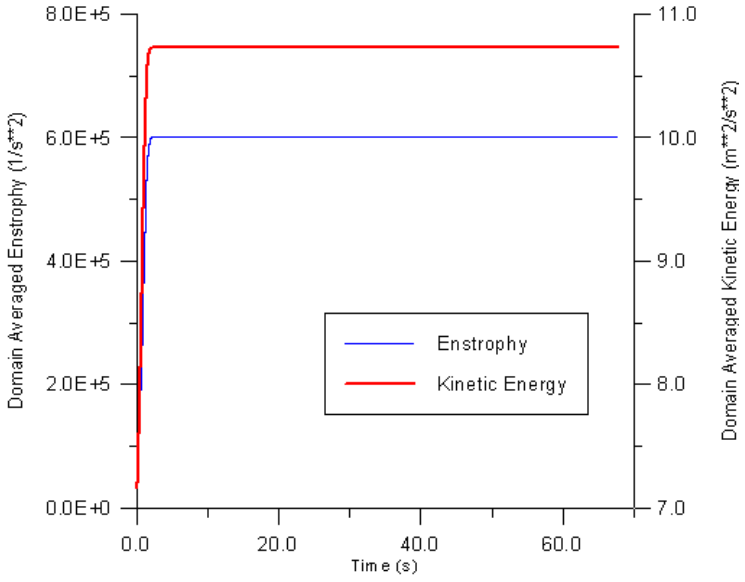

Figure 1. Temporal evolution of domain averaged enstrophy and kinetic energy.

Figure 2 shows the vertical profiles of $\overline{u}$ normalized by the centerline longitudinal component of the velocity U_c, at positions x = 0.0 m (channel entrance), x = 2.5, and x = 4.0 m. It is apparent a change in the profile shape along the x-direction as it evolves from the laminar profile at the inflow boundary to a fully developed turbulent situation near the channel exit region. Notice that the vertical gradient of $\overline{u}/U_c$ near the wall is significantly larger at $x = 0.8L$ = 4.0 m than at the channel entrance. Such a change is consistent with a laminar-turbulent transition of the flow.

Schlichting *et al.* (2000) state that in a laminar-turbulent transition boundary layer on a flat plate there is a significant reduction in the shape factor $H_{12} \equiv \delta_1/\delta_2$, where δ_1 is the displacement thickness and δ_2 is the momentum thickness. According to those authors experimental results showed that for a plate boundary layer, in the laminar region H_{12} = 2.59, which decreases to a value of approximately 1.4 in the turbulent region. Figure 3 displays the longitudinal profile of H_{12} obtained in the numerical simulation with the LSM method. The graph shows that at the channel entrance

$H_{12} \approx 2.5$, and its value gradually decreases along the x-direction, reaching approximately 1.55 at x = 2.6 m. The simulated spatial evolution of H_{12} along the x-direction is therefore consistent with that expected for a flow that undergoes a laminar-turbulent transition.

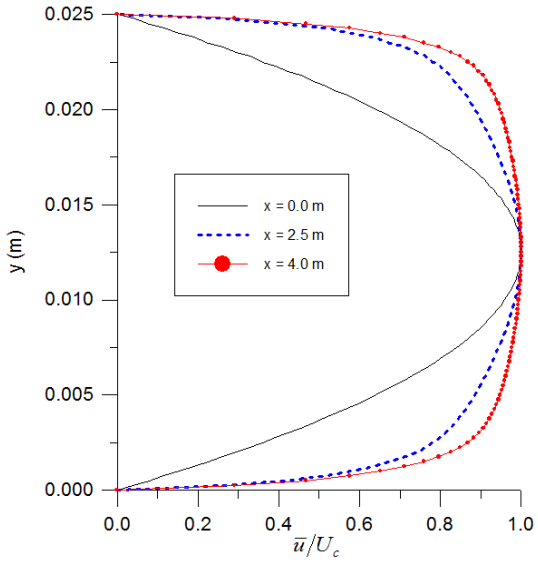

Figure 2. Vertical profiles of the normalized mean longitudinal velocity component $\overline{u}/U_c$.

Kim *et al.* (1987) calculated statistics of flow variables in their simulation of a low Re ($Re_\tau = 180$) turbulent channel flow. Those statistics were averages taken along the x and y directions, because periodic boundary conditions were employed. On the other hand in the LSM simulation the statistics vary along the x-direction, because physical boundary conditions were employed. In order to be able to compare the results of those two studies, the statistics referent to position x = 2.5 m of the LSM simulation were chosen, because at that position the ratio $U_c/U_m = 1.16$, which was the characteristic value of the simulation of Kim *et al.* (1987). Table 1 presents the comparison of the statistics of Kim *et al.* (1987) with those of the LSM simulation. It can be seen that the results of the two simulations agree well.

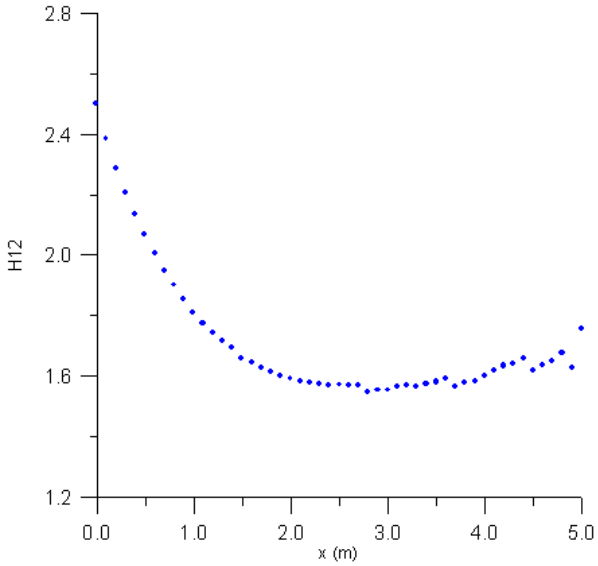

Figure 3. Longitudinal profile of shape factor H_{12}.

An important aspect to be analyzed in a simulation of a channel flow is the ability to represent the "law of the wall" in the vicinity of the solid boundary. Figure 4 presents the vertical profiles of u^+ at positions x = 2.2 m, x = 2.5 m, x = 2.7 m x = 3.0 m, and x = 3.2 m obtained in the LSM simulation. The functions of the "law of the wall" and the original profile of u^+ obtained by Kim *et al.* (1987) are also represented in the same figure. The LSM simulated profiles showed

a good agreement among themselves as well as with the functions of the "law of the wall". Notice that no adjustment was applied to the profiles to fit them to the "law of the wall". Another important point is the fact that the first grid point adjacent to the channel wall was located at $y^+ = 4.7$. That value is about 90 times larger than the minimum grid spacing of the DNS study of Kim *et al.* (1987) and almost 10 times larger than the minimum grid spacing of the LES channel flow simulation of Hughes *et al.* (2001) . It is noteworthy that the LSM method does not employ any turbulence model or parameterization to represent small scale effects not resolved by the grid. Only the aggregated N-S equations are used in the numerical simulation.

Table 1. Comparison of flow variables statistics.

	Kim *et al.*	LSM simulation $x = 2.5$ m	Relative difference (%)
Re_τ	180	237	-
U_c/U_m [(1)]	1.16	1.16	-
U_m/u_τ	15.63	15.61	- 0.13
U_c/u_τ	18.20	18.12	- 0.44
C_{f0} [(2)]	6.04×10^{-3}	6.09×10^{-3}	+ 0.83
δ_1/δ	0.141	0.138	- 2.13
δ_2/δ	0.0876	0.088	+ 1.15
H_{12}	1.62	1.57	- 3.09

(1) : $U_m \equiv \left(1/\delta \int_0^\delta \bar{u} dy \right)$; (2) : Skin-friction coefficient

Another point to be analyzed is the ability of the simulation to correctly represent the energy spectrum of the turbulent flow. Figure 5 presents the energy spectra of the u-velocity component at positions $x = 2.5$ m and $x = 4.0$ m, in the centerline of the flow ($y = \delta$). The spectra have a slope of approximately -3 in log-log representation in a range of frequencies spanning more than one decade. According to Kraichnan (1967), in two-dimensional turbulence such spectral regions correspond to inertial subranges characterized by direct vorticity cascade.

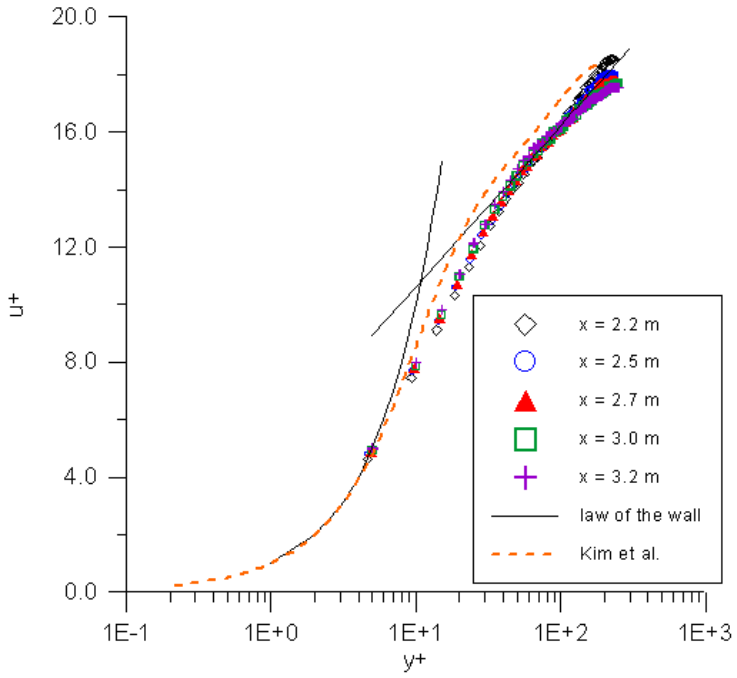

Figure 4. Vertical profiles of normalized horizontal velocity component u^+.

In channel flows viscous effects prevail in the near-wall region and turbulent effects are more important away from the solid boundaries. In the LSM simulation the same system of equations was employed for all parts of the domain, without any *ad hoc* corrections or adjustments, demonstrating that the LSM method is valid both for laminar and turbulent flows

It is known that spatial interpolations in the semi-Lagrangian method introduce dissipation into a numerical simulation. It can be therefore questioned if the turbulent mixing and viscous dissipation processes reproduced by the LSM simulation are mostly due to numerical dissipation effects introduced by the interpolations. An experiment then

was made to clarify this point. The simulation was run for the exact same conditions of the previous experiment, but in this case the viscosity term of the N-S equations was "switched-off" at time $t = 9.69$ s, by setting the kinematic viscosity $\nu \approx 0$ (actually 10^{-36}), all other conditions remaining unchanged. This new flow would be therefore inviscid. If the simulated small scale effects (turbulent mixing and viscous dissipation) are mostly due to numerical dissipation of the spatial interpolations it should not be expected a significant change in the characteristics of the simulated flow. On the other hand, in the case a remarkable change occurs in the conditions of the numerical simulation, it will have been demonstrated that the turbulent and viscous processes simulated by the LSM method are actual physical responses to the viscosity term in the N-S aggregated equations.

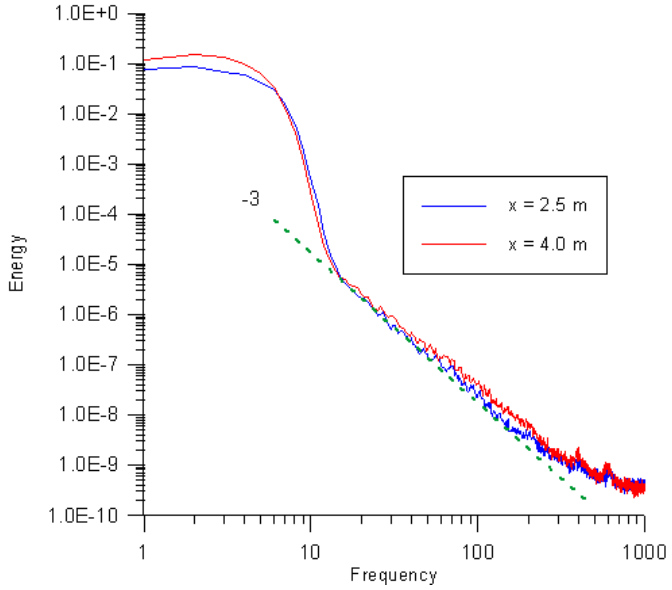

Figure 5. Energy spectra. Units for frequency and energy are arbitrary.

Figure 6 presents the temporal evolutions of the domain averaged enstrophy and kinetic energy for the simulation where the viscosity was "switched-off" at $t = 9.69$ s. It is apparent that as ν is set ≈ 0 there was an immediate response in the simulation, with increase both in the kinetic energy and the enstrophy. Those effects are expected since there was neither viscous nor turbulent drag acting on the flow.

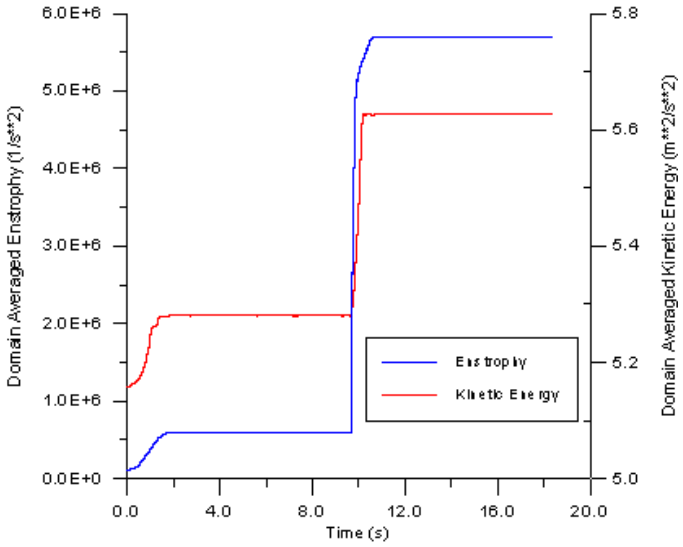

Figure 6. Temporal evolution of domain averaged enstrophy and kinetic energy.

The energy spectra would also give valuable information on the flow characteristics. Figure 7 shows the (inviscid) energy spectra at the same two coordinates whose spectra for the previous (viscous) simulation were presented in Fig. 5. It can be seen that these new spectra are remarkably different from the previous ones. There is no evidence of inertial

subranges and the spectra in the high frequency range are almost flat. Such spectra resemble those of random processes, which could be related to numerical random noise.

The results of the second simulation demonstrate that the small scale turbulent and viscous effects reproduced by the LSM simulation are actual physical processes related to the viscous term in the aggregated N-S equations and not produced by numerical dissipation due to spatial interpolation of the semi-Lagrangian method. Those processes are due to the mixing in the smaller scales represented by the LSM operator, and numerically reproduced by the averaging property of the finite differences method.

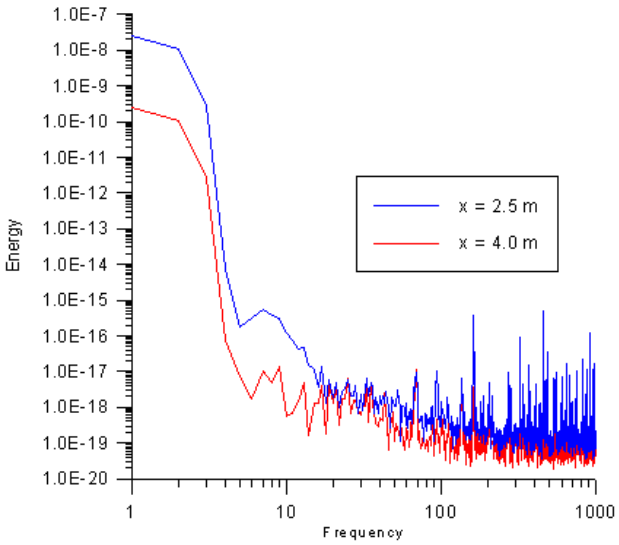

Figure 7. Energy spectra. Units for frequency and energy are arbitrary.

5. CONCLUSION

This paper introduced the Lagrangian Sample Mean method for numerical simulation of viscous fluid flows. The method is based on the hypothesis that small scale processes are responsible for mixing of fluid properties within inifinitesimal fluid parcels in the Lagrangian frame of reference. The homogenization of the fluid properties produced by the mixing is represented by a linear operator, and the resulting mean is called Lagrangian Sample Mean (LSM).

Application of the LSM operator to the N-S equations written in the Lagrangian frame of reference produced a system that had the same functional form of the original one. Since there were no new terms, the number of unknowns was the same as the number of equations. Therefore, there was no closure problem, and the resulting system could be solved without the use of closure models or parameterizations.

The method was applied to simulate a two-dimensional turbulent channel flow, by using the semi-Lagrangian method associated to spatial discretization by finite differences. The former method meets the requirement of remaining in the Lagrangian frame of reference, whereas the latter has the property of averaging variables within the grid cells.

The simulation was able to reproduce the laminar-turbulent transition of the flow, the velocity profile of the “law of the wall”, and significant inertial subranges characteristic of direct vorticity cascade in two-dimensional turbulent flows. Flow statistics of the LSM simulation showed a good agreement with results of DNS channel flow studies available in the literature as well.

A second simulation was made to evaluate if the small scale effects reproduced by the LSM method were mostly due numerical dissipation introduced by spatial interpolations of the semi-Lagrangian method. The results showed that the LSM method produces actual physical responses to the viscous term in the N-S equations, and that the simulated turbulent and viscous effects are not just a fortuitous product of the numerical dissipation.

The good agreement of the LSM simulation with experimental and theoretical results demonstrated that the underlying hypothesis of the LSM method is correct. Although more work is needed to validate the method in other flow conditions, the results presented in this study indicate that the LSM method can give a significant contribution to the field of computational fluid dynamics.

6. REFERENCES

Aris, R., 1962, “Vectors, Tensors, and the Basic Equations of Fluid Mechanics”, Prentice-Hall Inc., Englewood Cliffs, USA, 300 p.

Devore, J.L., 1987, “Probability and Statistics for Engineering and the Sciences”, Brooks/Cole Publishing Company, Monterey, USA, 672 p.

Hughes, T.J.R., Oberai, A.A. and Mazzei, L., 2001, "Large Eddy Simulation of Turbulent Channel Flows by the Variational Multiscale Method", Physics of Fluids, Vol. 13, No. 6, pp. 1784-1799.

Kim, J., Moin, P and Moser, R., 1987, "Turbulence Statistics in Fully Developed Channel Flow at Low Reynolds Number", Journal of Fluid Mechanics, Vol. 177, pp. 133-166.

Kraichnan, R.H., 1967, "Inertial Ranges in Two-Dimensional Turbulence", Physics of Fluids, Vol. 10, No. 7, pp. 1417-1423.

Kundu, P.K., 1990, "Fluid Mechanics", Academic Press Inc., San Diego, USA, 638 p.

Pope, S.B., 2000, "Turbulent Flows", Cambridge University Press, Cambridge, United Kingdom, 771 p.

Robert, A., Yee, T.L., and Ritchie, H., 1985, "A Semi-Lagrangian and Semi-Implicit Numerical Integration Scheme for Multilevel Atmospheric Models", Monthly Weather Review, Vol. 113, pp. 388-394.

Schlichting, H., Gersten, K., Krause, E. and Oertel JR., H., 2000, "Boundary Layer Theory", 8 ed., Springer-Verlag, Berlin, Germany, 799 p.

Stull, R.B., 1988, "An Introduction to Boundary Layer Meteorology", Kluwer Academic Publishers, Dordrecht, The Netherlands, 666 p.

Wilcox, D.C., 2000, "Turbulence Modeling for CFD", 2 ed., DCW Industries, Inc., Anaheim, USA, 540 p.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.