FREE VIBRATION ANALYSIS OF A PRESTRESSED HYPERELASTIC CIRCULAR MEMBRANE

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Abstract. *Membranes have been used in several engineering branches. Applications include inflatable fluid containers, large roof structures and many problems in biomedical engineering. In this type of problem, geometric and material non-linearities play an important role and usually large deformations must be taken into account. Static analysis of membranes under several loading conditions has been conducted by several authors in the past. On the other hand, research on the dynamic response of membranes is rare in literature. In this work the axisymmetric vibrations of a circular pre-stressed membrane is analyzed in detail. The material of the membrane is assumed to be neo-Hookean, isotropic and incompressible. Based on the theory of finite deformations of hyperelastic membranes, a variational formulation of the problem is developed and the equations of motion are obtained through the principle of the stationary potential energy. Then a modal expansion that satisfies the relevant boundary and continuity conditions is used together with the Galerkin method to discretize the equations in space. The obtained natural frequencies and modes are compared with the results obtained by the finite element method through the computational program ABAQUS and the results from the classical wave equation.*

Keywords*: circular membrane, free vibrations, hyperelastic material, large deformations.*

1. INTRODUCTION

The analysis of membranes under large deformations is based on the non-linear elasticity theory. Usually, the solution procedure follows the pioneering work of Green and Adkins (1960). Some research has been developed in this field, many of them related with the equilibrium and stability of membranes submitted to internal pressures and loads acting along one of the edges (Ratner, 1982; Pamplona and Bevilacqua, 1992; Haugthon, 2001; Mockensturm and Golgbourne, 2006). For sufficiently simple geometries and loads, analytical solutions can be found in literature. However, in the majority of these studies numerical methods have been used.

In addition to the theoretical and numerical contributions, some experimental studies of the equilibrium and stability of membranes exist. These experimental studies can assist in the choice of the more adequate constitutive model and in the identification of certain non-linear phenomena characteristic of the type of structure (Pamplona et al. 2001; Pamplona et al., 2006).

Recently Selvadurai (2006) analyzed experimentally a hyperelastic membrane and modeled the material using the Mooney-Rivlin, neo-Hookean, Blatz-Ko, Yeoh and Ogden constitutive relations. These constitutive models were used in the numerical study of the transversal displacements of circular hyperelastic membrane. Later, the author compared the experimental results with the results obtained through a computational model using the finite elemt program ABAQUS.

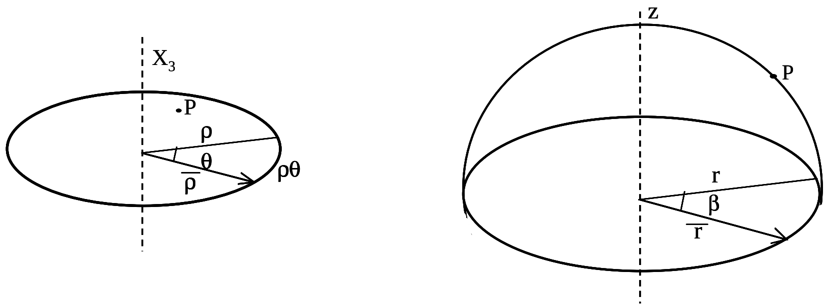
These previous studies dealt with membranes under static loads. The dynamic behavior of hyperelastic membranes is not so common in literature. Among the few contributions in this field we can mention the works by Akyüz and Ertepinar (1997, 1999) and Mockensturm and Goulbourne (2006). In both the cases, the membrane is first tensioned and later its dynamic behavior is analyzed.

Thus, the aim of the present work is to investigate the dynamic behavior of plain circular membranes under traction along the boundary. The material is considered to be neo-Hookean, isotropic and incompressible.

For this, a variational formulation is developed for the solution of a circular membrane, with or without a central hole, subjected to a prescribed extension in the radial direction along the outer boundary. The equations of motion of the membrane are derived through the principle of the stationary potential energy. The usual ordinary differential equations of motion are obtained by discretizing the membrane in space through the Galerkin method, using an expansion that satisfies the relevant boundary and continuity conditions. The free vibration modes and the natural frequencies of these membranes are obtained by the solution of the linearized equations of motion. These results are compared with the results obtained by finite elements through the FE program ABAQUS 6.5.

2. PROBLEM FORMULATION

Consider an undeformed, plain circular membrane with or without an internal hole, of thickness H and radius $\overline{r}$, submitted to a uniform radial extension w_t and a perturbation in the transversal direction, as illustrated in Fig. 1.



(a) Undeformed (b) Deformed

Figure 1 – Deformed and undeformed configurations of the circular membrane.

Thus, considering the system of co-ordinates (r, q), shown in Fig. 1a, the co-ordinates of a generic point P in the undeformed configuration are given by:

$$X_1 = r\cos(q) \\ X_2 = r\sin(q) \\ X_3 = X_3 \tag{1}$$

where X_1 is the radial co-ordinate of the undeformed membrane, X_2 is the co-ordinate in the circumferential direction and X_3 is the co-ordinate in the transversal direction of the undeformed membrane.

After the deformation, due to the radial displacement and the transversal disturbance, the co-ordinates of a generic point P in the deformed configuration are:

$$y_1 = r(r,q)\cos b(r,q) \\ y_2 = r(r,q)\sin b(r,q) \\ y_3 = z(r,q) \tag{2}$$

where y_1 is the co-ordinate in the radial direction of the deformed membrane, y_2 is the co-ordinate in the circumferential direction and y_3 is the co-ordinate in the transversal direction.

In relation to the system of co-ordinates (r, q) the contravariant and covariant metric tensors of the undeformed membrane is given by:

$$a_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad a^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1/r^2 \end{bmatrix} \tag{3}$$

$$a = \det(a_{ij}) = r^2 \tag{4}$$

and, for the deformed membrane, the covariant metric tensor is:

$$A_{ij} = \begin{bmatrix} r,_r{}^2 + r^2 b,_r{}^2 + z,_r{}^2 & r,_q\, r,_r + r^2 b,_q\, b,_r + z,_q\, z,_r \\ r,_q\, r,_r + r^2 b,_q\, b,_r + z,_q\, z,_r & r,_q{}^2 + r^2 b,_q{}^2 + z,_q{}^2 \end{bmatrix} \tag{5}$$

$$A = \det(A_{ij}) = (r,_r{}^2 + r^2 b,_r{}^2 + z,_r{}^2)\,(r,_q{}^2 + r^2 b,_q{}^2 + z,_q{}^2) - (r,_q\, r,_r + r^2 b,_q\, b,_r + z,_q\, z,_r)^2 \tag{6}$$

where $(\ ),_r = \dfrac{\partial(\ )}{\partial r}$; $(\ ),_q = \dfrac{\partial(\ )}{\partial q}$.

Due to the incompressibility of the material, the strain invariant $I_3 = 1$. So the principal stretch in the transversal direction is:

$$l_3{}^2 = \frac{a}{A} \tag{7}$$

Based on these expressions, the first strain invariant, I_1, can be written as:

$$\begin{aligned} I_1 &= r,_r{}^2 + r^2 b,_r{}^2 + z,_r{}^2 + \frac{r,_q{}^2 + r^2\, b,_q{}^2 + z,_q{}^2}{r^2} + \\ &+ \frac{r^2}{(r,_r{}^2 + r^2 b,_r{}^2 + z,_r{}^2)\,(r,_q{}^2 + r^2 b,_q{}^2 + z,_q{}^2) - (r,_q\, r,_r + r^2 b,_q\, b,_r + z,_q\, z,_r)^2} \end{aligned} \tag{8}$$

2.1. Equations of Motion

Considering that the membrane behaves as a neo-Hookean material, the strain energy density, W, can be described as a function of the first strain invariant as:

$$W = C_1(I_1 - 3) \tag{9}$$

where C_1 is a material parameter.

The elastic strain energy is obtained by the volumetric integral of W in the undeformed configuration, that is:

$$E = \int_{r0}^{\bar{r}} \int_{0}^{2p} h\, r\, W(r;q;r;r,_r;r,_q;z,_r;z,_q;b,_r;b,_q)\, dq\ dr \tag{10}$$

The Lagrange energy function is given by the difference between the kinetic and elastic energy of the system. By applying the usual tools of variational calculus, the three equations of motion of the membrane are:

$$-\frac{\partial}{\partial r}\left(\frac{\partial W}{\partial z,_r}\right) - \frac{\partial}{\partial q}\left(\frac{\partial W}{\partial z,_q}\right) + r\,\Gamma \frac{\partial^2 u}{\partial t^2} = 0 \tag{11}$$

$$-\frac{\partial}{\partial r}\left(\frac{\partial W}{\partial b,_r}\right) - \frac{\partial}{\partial q}\left(\frac{\partial W}{\partial b,_q}\right) + r\,\Gamma \frac{\partial^2 v}{\partial t^2} = 0 \tag{12}$$

$$r\,\frac{\partial W}{\partial r} - \frac{\partial}{\partial r}\left(\frac{\partial W}{\partial r,_r}\right) - \frac{\partial}{\partial q}\left(\frac{\partial W}{\partial r,_q}\right) + r\,\Gamma \frac{\partial^2 w}{\partial t^2} = 0 \tag{13}$$

where Γ it is the membrane material density.

As the membrane was first submitted to a static radial extension and then submitted to a general time-dependent disturbance, the displacement field of the circular membrane can be described as:

$$z(r,q,t) = z_o(r) + u(r,q,t) \tag{14}$$

$$r(r,q,t) = r_o(r) + w(r,q,t) \tag{15}$$

$$b(r,q,t) = b_o(r) + v(r,q,t) \tag{16}$$

where $w(r, q, t)$, $u(r, q, t)$ and $v(r, q, t)$ represent the time-dependent displacements in the radial, transversal and circumferential directions respectively; and $r_0(r)$, $z_0(r)$ and $b_0(r)$ are the co-ordinates of a point on the pre-tensioned membrane.

The static solution of the membrane under a uniform radial traction is obtained by solving the in-plane equilibrium equations by the shooting method. In this case the transversal and circumferential displacements are:

$$z_o(r) = 0 \tag{17}$$

$$b_o(r) = q \tag{18}$$

Based on the results obtained by the shooting method, the radius of the extended circular membrane without a central hole can be described by the following linear function of r:

$$r_o(r) = T\,r \tag{19}$$

where T is a constant equal to the external deformed radius of the membrane, while for the annular membrane the following function of r is obtained:

$$r_o(r) = A\ln r + B\,r^2 \ln r + C\,r^2 + Dr + E \tag{20}$$

where A, B, C, D and E are constants that depend on the properties of the deformed membrane.

Substituting equations (14) to (16) into the strain and kinetic energy and later in the equations of motion, (11) to (13), the equations of motion in terms of the displacements are obtained. Substituting Eq. (19) in Eq. (11), one arrives at the linearized equation of motion:

$$\frac{\partial^2 u}{\partial t^2} = \frac{2C_1}{\Gamma}\left(\frac{1}{T^6} - 1\right)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial q^2}\right) \tag{21}$$

which is similar to the classical wave equation if:

$$c^2 = \frac{2C_1}{\Gamma}\left(\frac{1}{T^6} - 1\right) \tag{22}$$

So the transversal displacement u can be obtained by solving Eq. (21) together with the relevant boundary conditions. The analytical solution for the circular membrane is obtained by separation of variable:

$$u(r,q,t) = A_m J_n\left(a_{mn}\frac{r}{\bar{r}}\right)\cos(nq)\,e^{(Iw\,t)} \qquad n = 0,1,2,\cdots \tag{23}$$

while for the annular membrane, one has:

$$u(r,q,t) = A_m Y_n\left((a_{mn} - a_{m+1n})\frac{(r + r^* - r_o)}{\bar{r} - r_o}\right)\cos(nq)\,e^{(Iw\,t)} \qquad n = 0,1,2,\cdots \tag{24}$$

where A_m are the modal amplitudes; J_n is the Bessel function of first type and order n; Y_n is the Bessel function of the second type and order n; m is the number of half-waves in the radial direction; n is the number of waves in the circumferential direction; α_{mn} is m-th positive root of $J_n(\alpha)$, $\bar{r}$ is the undeformed external radius of the membrane; r_0 is the undeformed internal radius of the membrane and r^* is the first value of r where the Bessel function Y_n is zero.

The radial displacement w and circumferential displacement v are very small compared with the transversal displacement u.

3. NUMERICAL RESULTS

Consider an undeformed circular membrane of radius $\bar{r}$ = 1 *m*, and variable thickness. The material of the membrane is considered to be neo-Hookean, isotropic and incompressible, with a constant $C_1 = 0.17$ *MPa* and density G=2200 Kg/m^3. For the annulus, the internal radius is $r_o = 0.20$ *m*

The initial displacement field for the circular membrane, considering two different values of the initial radial extension, w_t, are:

$$w_t = 0.10 \qquad r_o(r) = 1.1\,r \tag{25}$$

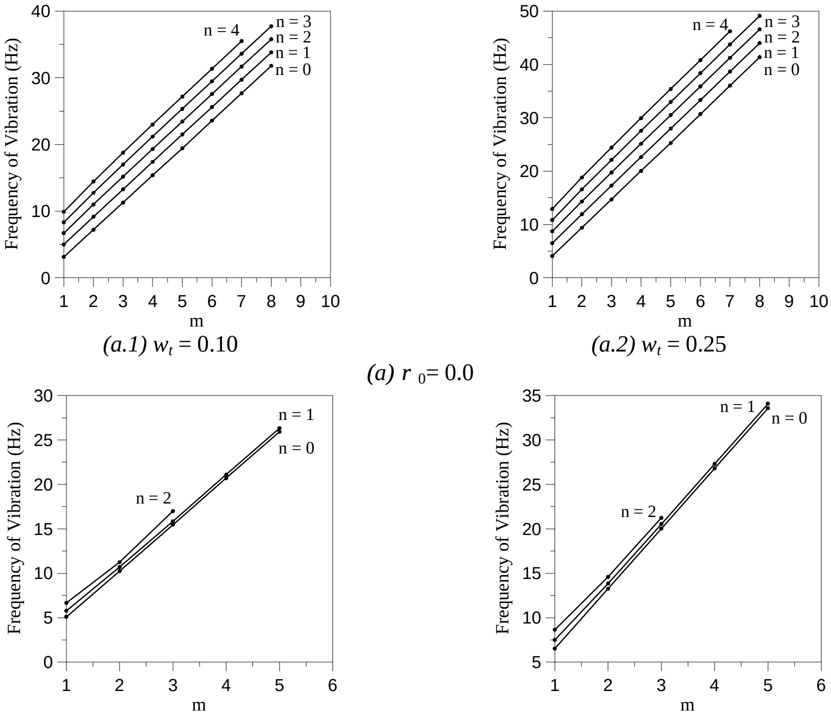
$$w_t = 0.25 \qquad r_o(r) = 1.25\,r \tag{26}$$

while for the annular membrane the following displacement field is obtained:

$w_t = 0.10 \qquad r_o(r) = 0.045 \ln r - 0.066\, r^2 \ln r + 0.111\, r^2 - 0.095 r + 0.084 \qquad (27)$

$w_t = 0.25 \qquad r_o(r) = 0.114 \ln r - 0.156\, r^2 \ln r + 0.268\, r^2 - 0.223 r + 0.208 \qquad (28)$

Fig. 2 illustrates the variation of the membrane circular vibration frequency as a function of the number of radial half-waves, *m*, for increasing values of the number of circumferential waves, *n*, considering the two prescribed values of w_t.



(a.1) w_t = 0.10 *(a.2)* w_t = 0.25

(a) r_0= 0.0

(b.1) w_t = 0.10 *(b.2)* w_t = 0.25

(b) r_0= 0.20

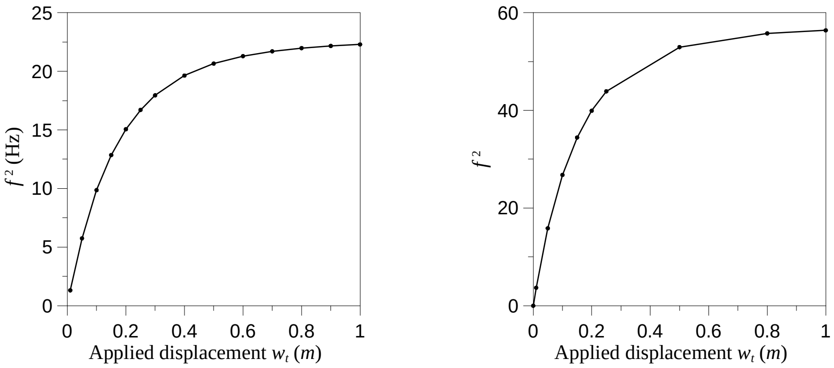
Figure 2 - Vibration frequencies.

The results found for the five first frequencies of the circular membrane are presented in Tab. 1.

Table 1. Frequency of vibration (Hz)

$r_o = 0.0$				$r_o = 0.20$			
n	*m*	w_t = 0.10	w_t = 0.25	*n*	*m*	w_t = 0.10	w_t = 0.25
0	1	3.1401	4.0871	0	1	5.1082	6.5302
1	1	5.0032	6.512 2	1	1	5.7818	7.5020
2	1	6.7057	8.7282	2	1	6.6696	8.6591
0	2	7.2078	9.3817	3	1	7.8266	10.1615
3	1	8.3308	10.8434	4	1	9.10744	11.8361

It is observed through Fig. 2 and Tab. 1 that the frequency increases linearly with *m* and *n*. It also increases as the initial tension increases. So, the lowest natural frequency occurs for *m* = 1 and *n* = 0. The influence of *n* on the natural frequencies of the annular membrane is smaller than for a complete membrane. However, the general behavior remains the same. The influence of the initial traction on the frequencies can better be observed in Fig. 3 that it illustrates the variation of the square of the lowest vibration frequency in relation to the applied extension. Initially the frequency increases steadily up to very large initial extension. However, for large extensions, it converges to an upper bound.



$r_o = 0.0$ $r_o = 0.20$

Figure 3 - Frequency of vibration x applied traction displacement.

After that, the frequencies and the modes of vibration were calculated by the finite element method, using the computational program ABAQUS 6.5. The values of the vibration frequencies are then compared with the results found in the analytical solution.

For this solution the shell finite element S4R is used. First a convergence analysis of the mesh is conducted, leading to a mesh of S4R elements of length $L_{ele} < 0{,}028$. The meshes used for the circular membrane with and without a central hole are illustrated in Fig. 4.

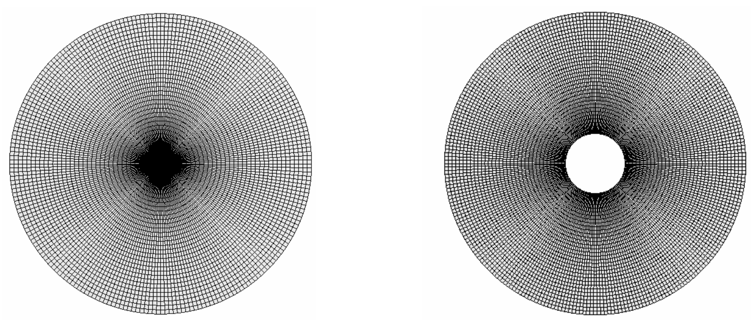


Figure 4 - The mesh used for the circular membrane.

The results for the circular membrane are shown in Tab. 2.

Table 2 - Frequency of vibration FEM (Hz)

$r_o = 0.0$							
$w_t = 0.10$				$w_t = 0.25$			
n	m	FEM Abaqus	semi-analytic	n	m	FEM Abaqus	semi-analytic
0	1	3.1397	3.1401	0	1	4.0866	4. 0871
1	1	5.0043	5.0032	--	--	5.6029	--
--	--	6.4134	--	1	1	6.5136	6. 512 2
2	1	6.7070	6.7057	--	--	7.5867	--

$r_o = 0.20$							
$w_t = 0.10$				$w_t = 0.25$			
n	m	FEM Abaqus	semi-analytic	n	m	FEM Abaqus	semi-analytic
0	1	5.1745	5.1082	0	1	6.6254	6.5302
1	1	5.6858	5.7818	1	1	7.3103	7.5020
2	1	6.9218	6.6696	--	--	8.3848	--
--	--	8.3865	--	--	--	8.4786	--

One can observe through the results presented in Tab. 2 a better agreement between the analytical vibration frequencies and those obtained through Abaqus, for the circular membrane without a central hole than for the annular one. A reason for this better agreement must be that in the first case the functions used to describe the transversal displacements (u) and the extended membrane (r_0) are exact, while in the second case they are only an approximation of the displacement field.

One interesting point is that the FEM results show certain frequencies and mode shapes which are not detected in by the analytical solution. These mode do not have the expected symmetries found in the analytical modes.

Its also important to notice that the results are in this case independent of the undeformed membrane thickness.

The numerical obtained vibration modes for the circular membrane with and without a central hole are illustrated in Fig. 5.

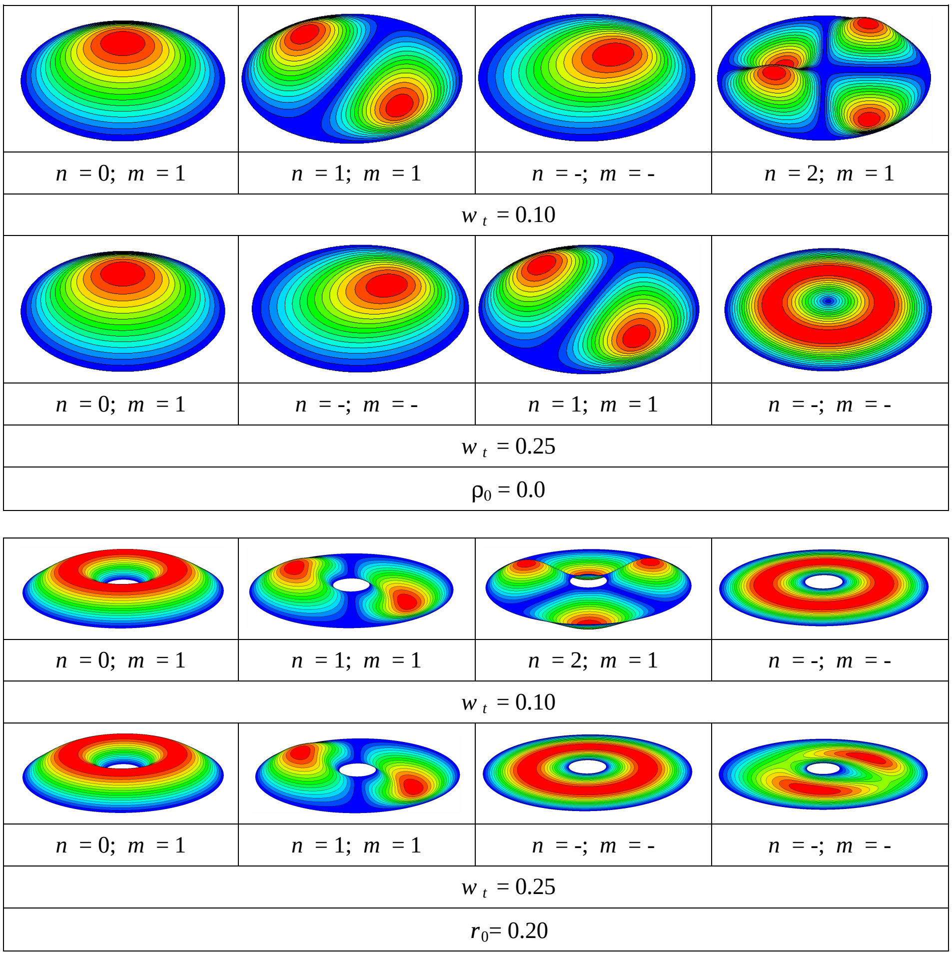


Figure 5 – Selected numerical vibration modes of the circular membrane.

4. CONCLUSIONS

The mathematical modeling of the large deformations of a circular membrane subjected to an initial radial extension and a subsequent time-dependent perturbation is presented in this paper. The membrane is considered to be isotropic, homogeneous and incompressible and that it can be described as a neo-Hookean material. Based on the expression of the strain energy density and applying the tools of variational calculus, the non-linear equations of motion of the pre-stressed membrane are derived. The linearized equations are than solved and the vibration frequencies and associated vibration modes are obtained. The linearized equation for the transversal displacement is reduced to a PDE similar to the classical wave equation. The analytical results are compared with those obtained by the finite element method using the software ABAQUS. Circular membranes with or without a central hole are considered in the analysis. A parametric analysis shows the influence of the initial radial extension on the vibration modes and frequencies. It is also observed that, with the increase of the applied traction, the values of the vibration frequencies increase in a non-linear manner converging to an upper bound for very large deformations. Some non-symmetric modes were obtained by the FEM program, which were not identified by the analytical solution, which predicts only modes with a certain number of uniform waves in the circumferential direction and a number of waves in the radial direction compatible with the roots of the associated Bessel function.

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