DYNAMIC INSTABILITY OF CYLINDRICAL SHELLS SUBJECTED TO SUDDEN STEP LOADS

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Abstract. *In this work the nonlinear behavior of a simply supported cylindrical shell under a sudden axial step load of infinite duration is investigated. The shell is modelled through the nonlinear shell theory of Donnell and the discretized equations of motion are obtained by the method of Galerkin based on a low dimensional modal solution derived in previous publications by the authors. The shell is initially considered to be at rest inside a potential well associated with the stable pre-buckling configuration. The effects of small initial geometric imperfections on the shell behavior is presented. The variation of the dynamic buckling load as a function of the geometric imperfection using Budiansky's stability criterion is evaluated. It is observed that the escape occurs in the neighborhood of the saddle point that defines the frontier of the stable region. The erosion of the safe region of the basin of attraction with the load increment is also investigated. The sudden reduction of the safe region highlights the imperfection sensitivity of the shell under this type of load.*

Keywords: cylindrical shells, dynamic buckling, basins of attraction, step load.

1. INTRODUCTION

In spite of its simple geometric form, cylindrical shells can display a complex behavior due to its geometric non-linearity and sensitivity to the initial geometric imperfections. Thus, the detailed knowledge of its static and dynamic behavior becomes essential to have a structural project that is simultaneously economic and safe. However little it is known on the nonlinear behavior of thin cylindrical shells under dynamic loads.

The influence of geometric imperfections on the static behavior of cylindrical shells under axial loads is well known in the technical literature, having been studied by various researchers in the past (Yamaki, 1984). These studies show that the imperfections have a great influence on the nonlinear behavior of the shell, causing a marked decrease in the critical load of the structure. However, the consideration of the initial geometric imperfections in the analysis of cylindrical shells under dynamic loads is inexpressive in literature. Del Prado (2001), using the nonlinear theory of Donnell for shallow shells, studied, for a cylindrical shell submitted to harmonic axial loads, its nonlinear vibrations and the dynamic instability. The author evaluated the effect of initial geometric imperfections on the static and dynamic behavior of the shell, concluding that small initial geometric imperfections in the form of the critical mode or the nonlinear mode can cause significant changes in its nonlinear dynamic behavior and in the critical loads.

Gonçalves et. al. (2005), using the nonlinear theory of Donnell and a reduced modal expansion including only two terms, have studied the effect of the geometric imperfections on the global stability of a simply supported cylindrical shell with internal fluid. To describe the initial geometric imperfections, two analyses have been made: first, a modal expansion with only the basic mode was assumed, and, in the second one, a modal solution containing the basic mode plus the axi-symmetrical mode. It is observed that the results are sensitive to the form of the imperfection. They also noticed that the geometric imperfections did not change the mechanisms of loss of stability of the shell, as observed by Silva et. al. (2004a, 2004b). The presence of initial geometric imperfections diminishes the safe region of the basin of attraction of the pre-buckling solution as the level of imperfection increases.

In the present work, the nonlinear theory of Donnell for shallow shells together with a low dimensional model is used to study the nonlinear vibrations and the stability boundaries of a cylindrical shell submitted to a sudden axial load. The nonlinear differential equations of motion are solved by the Runge-Kutta method. The main objective of this study is to study the influence of the geometric imperfections and a suddenly applied load of infinite duration (step load) on the global stability of the structure.

2. PROBLEM FORMULATION

A simply-supported cylindrical shell of radius R, thickness h and length L is considered. The shell material is defined by a modulus of elasticity E, a coefficient of Poisson n and a density r. The axial, circumferential and radial coordinates are defined, respectively, by x, y and z and the corresponding displacements by u, v and w, as shown in Fig. 1.

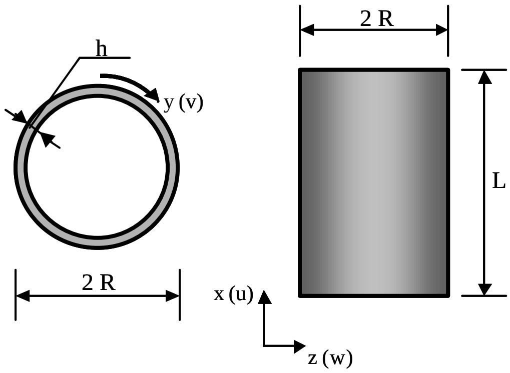


Figure1. Geometry of the shell and coordinate system.

The shell is submitted to an axial compressive load distributed along the upper and lower edges of the form:

$$P(t) = P_0 + P_1, \quad t \geq 0 \tag{1}$$

where P_0 is an initial static load and P_1 is the suddenly applied axial load of infinite duration.

Based on the nonlinear theory of Donnell for shallow shells, the nonlinear equations of motion in the transversal direction and the compatibility equation are given, respectively, by:

$$r\, h \ddot{w} + b_1 \dot{w} + b_2 \nabla^4 \dot{w} + D \nabla^4 w = F_{,yy}\left(w_{d,xx} + w_{i,xx}\right) + F_{,xx}\left(w_{d,yy} + w_{i,yy} + \frac{1}{R}\right) - 2F_{,xy}\left(w_{d,xy} + w_{i,xy}\right) \tag{2}$$

$$\frac{1}{E\,h}\nabla^4 f = \left[-\frac{1}{R} w_{d,xx} + \left(-w_{i,xx} w_{d,yy} - w_{d,xx} w_{i,yy}\right) + 2 w_{i,xy}\, w_{d,xy} + w_{d,xy}^2\right] \tag{3}$$

where:

$$F = f^F + f \qquad f^F = -\frac{1}{2} P_0 y^2 - \frac{1}{2} P_1 y^2 \tag{4}$$

where w_i is the initial geometric imperfection, ∇^4, the bi-harmonic operator, b_1 and b_2, the coefficients of viscous and material damping, respectively, and D, the flexural stiffness defined as:

$$D = E\,h^3 / 12(1 - n^2) \tag{5}$$

The following non-dimensional parameters are used in the analysis:

$$W = \frac{w}{h} \quad x = \frac{x}{L} \quad q = \frac{y}{R} \quad \Gamma_0 = \frac{P_0}{P_{cr}} = \frac{R\sqrt{3\left(1-n^2\right)}}{E\,h^2} P_0 \qquad \Gamma_1 = \frac{P_1}{P_{cr}} = \frac{R\sqrt{3\left(1-n^2\right)}}{E\,h^2} P_1 \tag{6}$$

where P_{cr} is the classical critical load for a simply-supported cylindrical shell (Brush and Almroth, 1975).

Previous studies on the proper modal solutions for cylindrical shells have shown that the modal solution must represent the existing coupling between asymmetric and axi-symmetric modes and describe in a consistent way the unstable post-critical response of the shell as well as the correct frequency-amplitude relation (Gonçalves and Del Prado, 2005). The most important modes are the basic buckling (or vibration) mode and the axi-symmetric mode with two times the number of half-waves in the axial direction as the basic mode. Thus, the lateral displacement can approximately be given by:

$$W_d = z(t)_{11} \cos(nq) \sin(mp\,x) + z(t)_{02} \cos(2mp\,x) \tag{7}$$

where n is the number of waves in the circumferential direction and m is the number of half-waves in the axial direction.

The initial geometric imperfections are taken in the form:

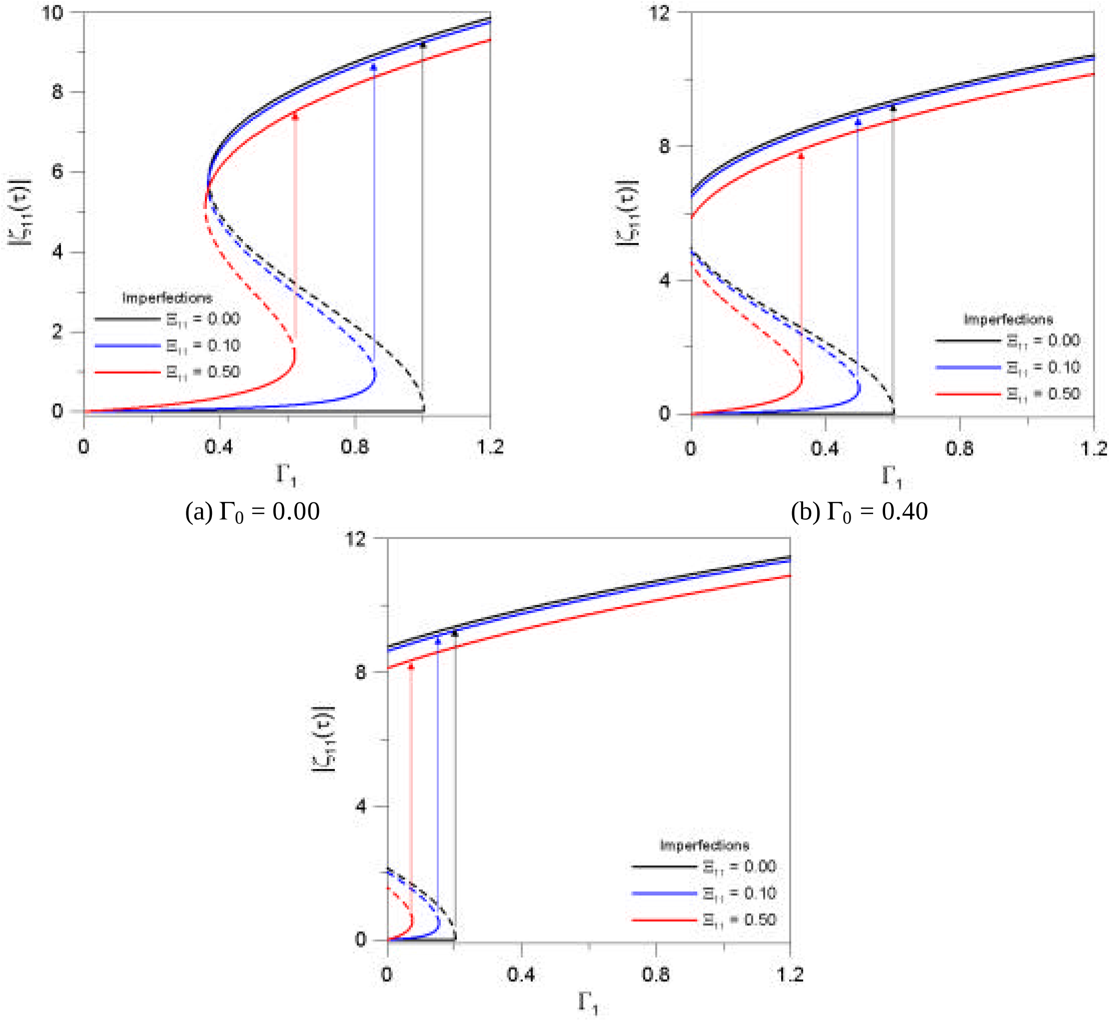
$$W_i = \Xi_{11} \cos(nq)\sin(mp\,x) \tag{8}$$

where Ξ_{11} is the imperfection amplitude.

Substituting the expansions for W_d and W_i on the right hand side of the compatibility equation, Eq. (3), it is possible to find an expression for the stress function f as a function of w_d and w_i that satisfies the in-plane boundary and continuity conditions of the shell. Now, substituting w_d, w_i and f in Eq. (2) and applying the Galerkin method, a set of discretized ordinary nonlinear equations of motion in terms of the time-dependent modal amplitudes $z(t)_{ij}$ is obtained.

3. NUMERICAL RESULTS

In this work a cylindrical shell of length L = 0. 4 m, radius R = 0.2 m and thickness h = 0.002 m is considered. The material of the shell is assumed as linear and elastic with modulus of elasticity E =2.1 x 10^{11} N/m² and coefficient of Poisson n = 0.30. The shell material density is r = 7850 kg/m³. The coefficients of viscous and material damping are: $b_1 = erh\omega_0$, with e = 0.0008 (Pellicano and Amabili, 2003), and $b_2 = h$D with h = 0.0001. For this geometry, the vibration mode associated with the minimum frequency has (m, n) = (1, 5).



(a) Γ_0 = 0.00

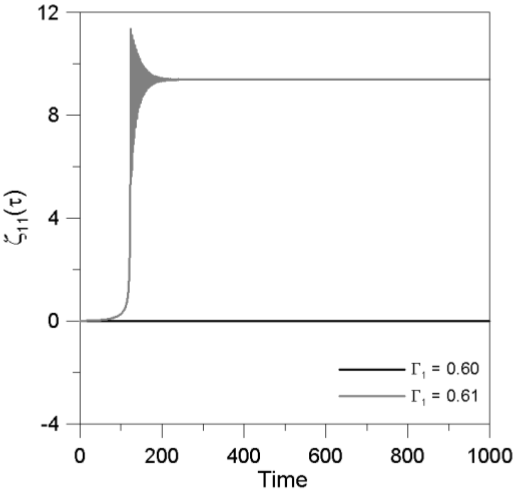
(b) Γ_0 = 0.40

(c) Γ_0 = 0.80

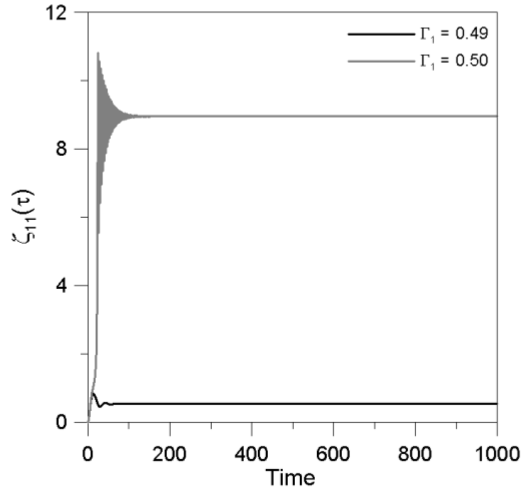
Figure 2. Variation of the modal amplitude z_{11} as a function of the step load parameter, Γ_1.

Fig. 2 shows the variation of the amplitude of displacement in the first mode, ζ_{11}, as a function of the non-dimensional step load parameter, Γ_1, for increasing values of the static pre-load level, Γ_0, and selected values of the initial geometric imperfection, Ξ_{11}. The shell response is obtained by solving the nonlinear equations by Newton-Raphson method together with continuation techniques, where the continuous curves represent the stable steady solutions and the dashed curves, the unstable solutions. After reaching the critical step-load value, the shell displays a sudden increase in the radial displacements, denoted by the arrows, jumping from a pre-buckling configuration to a post-buckling one characterized by large displacements.

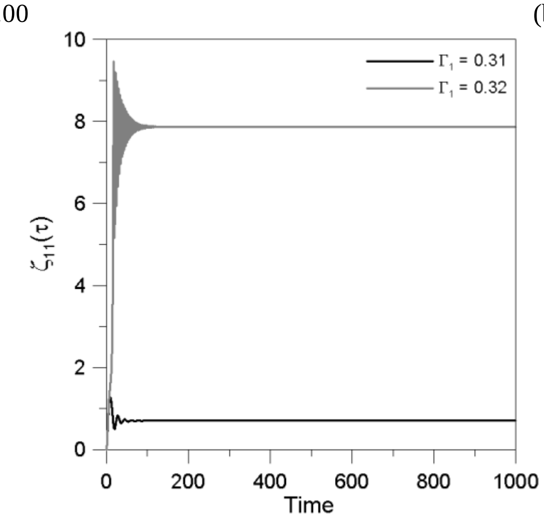
Fig. 3 shows the time response of the shell vibration amplitude $\zeta_{11}(t)$ for a static pre-load level of Γ_0 = 0.40 and increasing values of the imperfection amplitude Ξ_{11}, for values of Γ_1 just below and above the critical value. As observed in the figures, a small increment in the step load parameter causes a jump to a new equilibrium position along the advanced post-buckling path. The presence of initial geometric imperfection adds a certain amount of bending energy to the shell, so the initial response is no longer trivial as in the perfect case, as one can observe in Figures 3b and 3c for Γ_1 = 0.49 and Γ_1 = 0.31, respectively.



(a) Ξ_{11} = 0.00



(b) Ξ_{11} = 0.10



(c) Ξ_{11} = 0.50

Figure 3. Time response of the shell. Γ_0 = 0.40.

To evaluate this sensitivity to the initial geometric imperfections, the variation of the critical load, normalized by the critical load of the perfect shell as a function of the imperfection amplitude is investigated. The results are presented in Figure 4 for three different values of the static pre-load, Γ_0. The results show the high imperfection sensitivity of the shell under step load and that it increases with the amount of static pre-load. As shown in some previous paper by the authors (Silva et. al. 2004a, 2004b), the critical load of the imperfect system, although much lower than the critical load of the perfect one, is still an upper bound of the load capacity of the shell, due to its sensitivity to initial conditions. This sensitivity can be illustrated by the erosion of transient and permanent basins of attraction (Rega and Lenci, 2006).

The erosion of the basin of attraction of the permanent response with the increment of the step load parameter and the geometric imperfection was investigated and the results are shown in the Figs. 6-8. The dashed line represents the contour of the pre-buckling potential well in the plane $z_{11}(t) \, x \, z_{02}(t)$, which defines in this plane the set of initial conditions to be investigated in order to define the safe region. The safe region is defined as the set of all initial conditions whose time response, after a transient, converges to the safe permanent state. This is the black region in Figs.

6-8. The safe attractor is denoted by a yellow dot. The white region corresponds to the set of initial conditions that leads to escape from the pre-buckling well (dynamic buckling or snap-through).

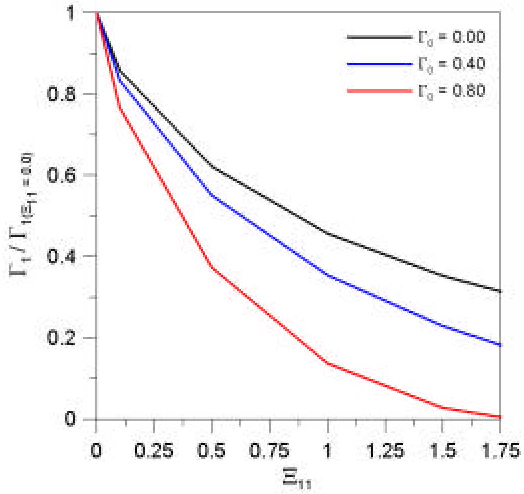


Figure 4. Variation of the normalized critical load with the imperfection magnitude.

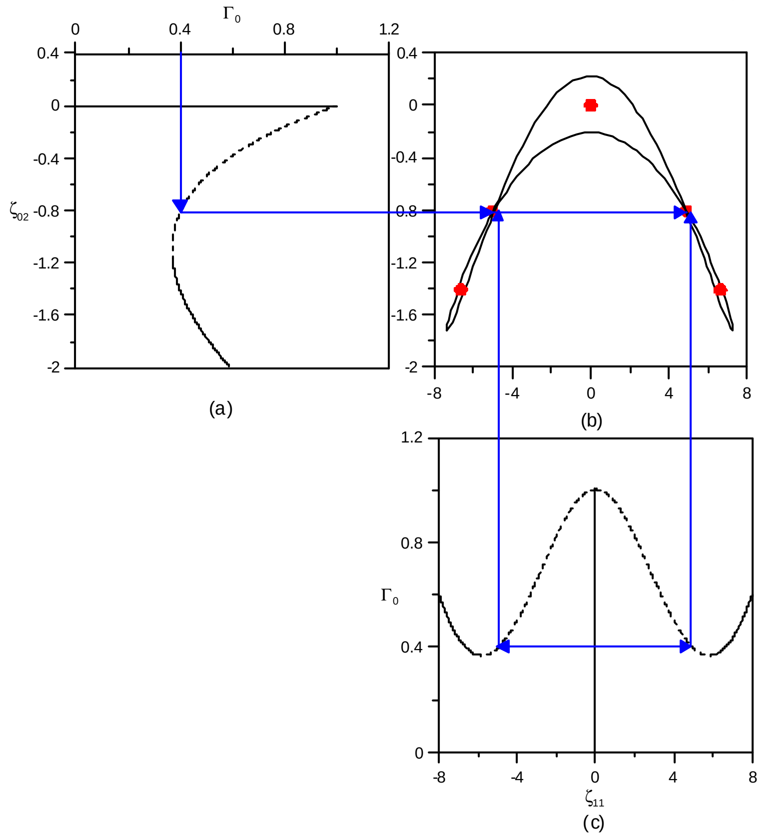
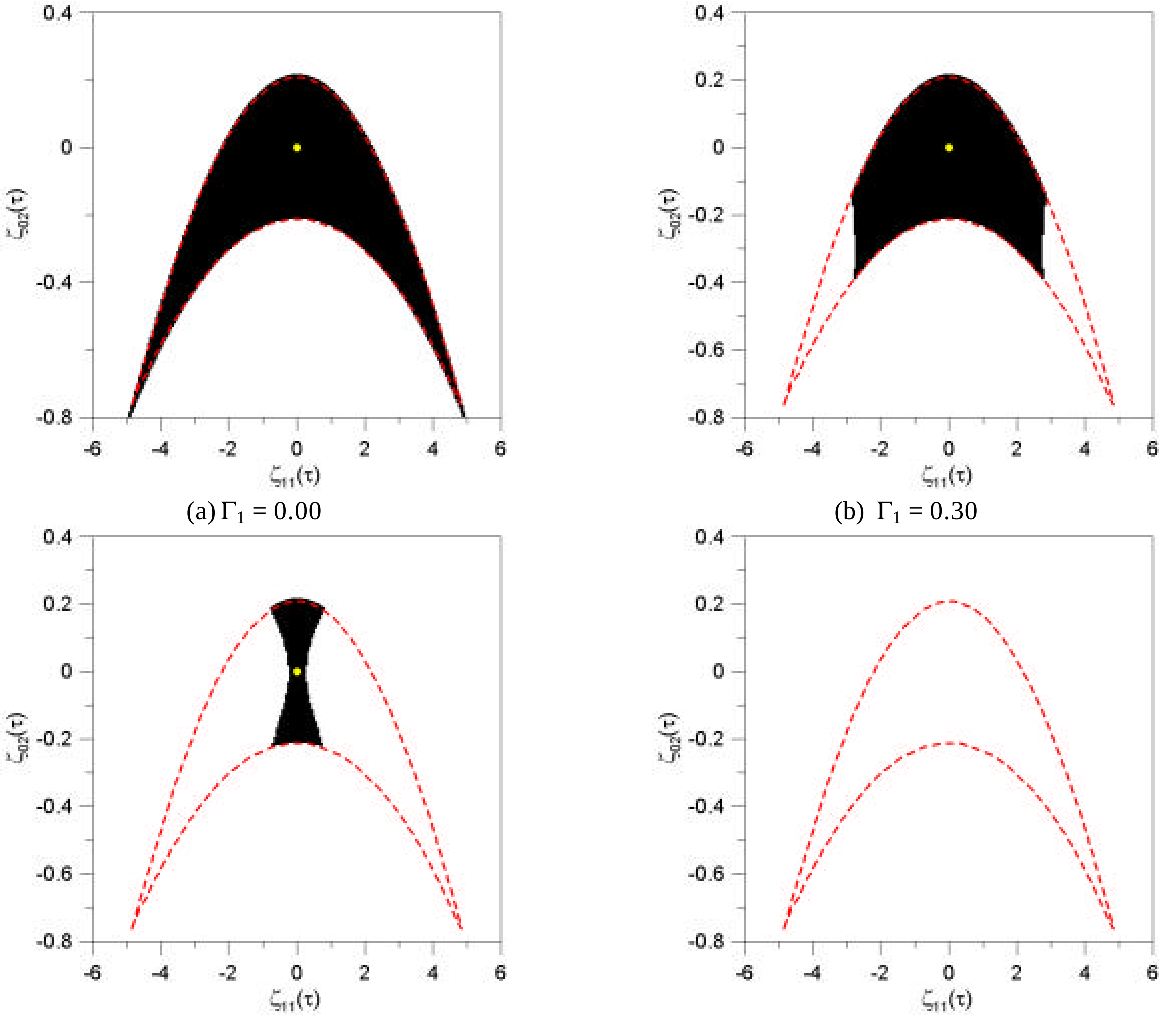


Figure 5. Projections of the post-critical response of the perfect shell (a and c) and the homoclinic and heteroclinic orbits connecting the two saddles (b) which define and four different classes of solution of the conservative system.

The region of inspected initial conditions is obtained from the analysis of the conservative system subjected to a static axial load. This gives a general idea of the global behavior of the shell (Gonçalves et. al. 2007a, 2007b). The profile of this region is explained by the initial post-critical behavior of the perfect shell depicted in Figs. 5a and 5c. At the critical point along the trivial pre-buckling path a sub-critical bifurcation occurs. The unstable post-buckling path then reaches a minimum value where a fold bifurcation occurs. The new emerging path is stable and it is connected with large bending deformations. For $\Gamma_0 < \Gamma_{min}$, the system presents only one equilibrium position. For $\Gamma_{min} < \Gamma_0 < \Gamma_{cr}$ the system has five equilibrium positions, three are stable and two are unstable. Table 1 show the eigenvalues associated to these equilibrium points, the unstable solutions are saddles. Thus, for a given value of Γ_0, the safe region is accurately defined by the heteroclinic orbits that connects the saddle points and bounds the trivial pre-buckling solution (Fig. 5b).

Table 1. Equilibrium points and the respective eigenvalues.

Equilibrium points	Coordinates $(z_{11}, z_{02}, \dot{z}_{11}, \dot{z}_{02})$	Eigenvalues	Type
P_1 (pre-buckling)	(0, 0, 0, 0)	±0.30î; ±3.168i	(center-center)
P_2, P_3 (saddle)	(±4.946, -0,811, 0, 0)	±0.255; ±3.527i	(saddle-center)
P_4, P_5 (post buckling)	(±6.632, -1,412, 0, 0)	±0.318i; 3.756i	(center-center)



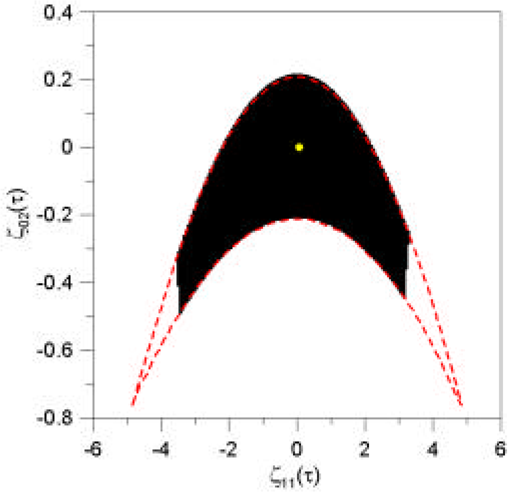
(a) $\Gamma_1 = 0.00$ (b) $\Gamma_1 = 0.30$

(c) $\Gamma_1 = 0.60$ (d) $\Gamma_1 = 0.61$

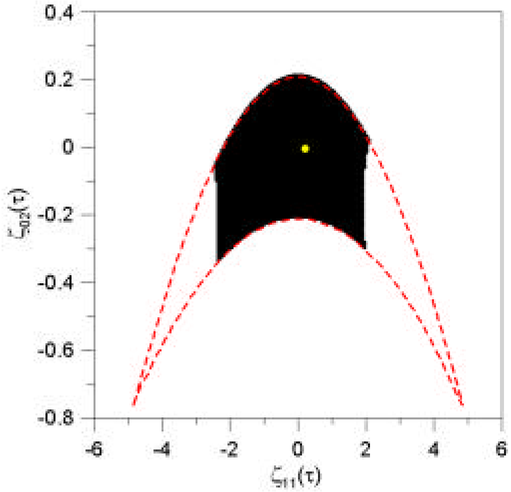
Figure 6. Erosion of the safe basin of attraction. Cross sections of the basins by the ζ_{11} x ζ_{02} plane for $\Gamma_0 = 0.40$ and $\Xi_{11} = 0.00$. $(\dot{z}_{11} = \dot{z}_{02} = 0.0)$.

Figures 6-8 show the erosion of the basin of attraction (black area) with the increment of the step load parameter, Γ_1, up to at the critical value where it becomes zero, that is, no set of initial condition on this plane converges to the desired solution. By comparing the basins of attraction of the perfect shell shown in Fig. 6 with those in Figs. 7 and 8, one can observe the marked influence of the initial imperfections on the topology of basin of attraction. With the increase of the geometric imperfections, the basin of attraction that was symmetrical in relation to co-ordinate ζ_{11}, losses its symmetry.

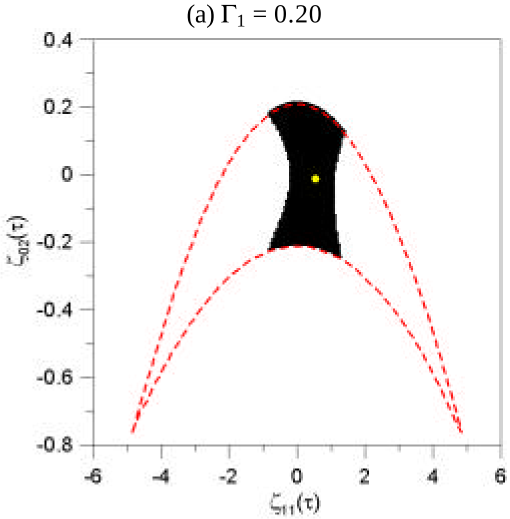
The influence of the imperfections on the process of erosion of the basin of attraction is illustrated in Fig. 9 that shows the variation of the safe area of the basin, normalized by the area of the conservative system (enclosed by the dashed curve), as a function of the step load parameter. The safe area decreases steadily as the load increases and becomes zero at the critical value of the load parameter. This decrease is accentuated by increasing imperfection. These results corroborate the high imperfection sensitivity of cylindrical shells under axial loads. Due to the unavoidable imperfections and disturbances from the initial conditions expected in real systems, the load capacity of the shell is expected to be much lower than the critical value, even when the imperfections are considered in the problem modeling.



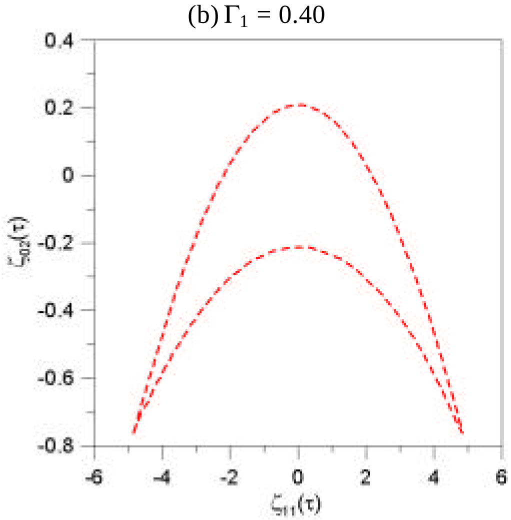
(a) $\Gamma_1 = 0.20$



(b) $\Gamma_1 = 0.40$

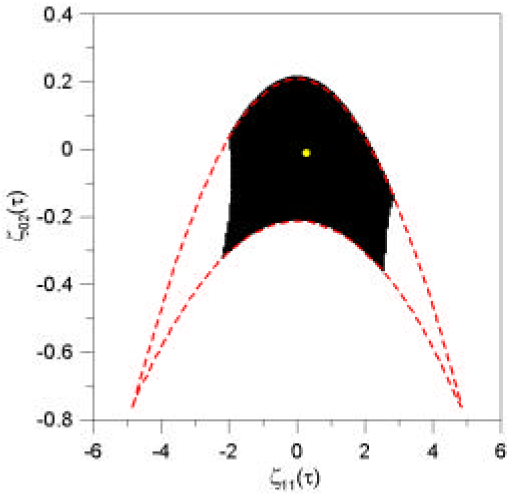


(c) $\Gamma_1 = 0.49$

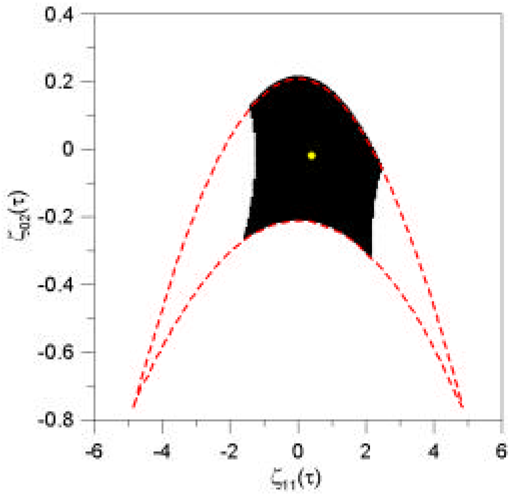


(d) $\Gamma_1 = 0.50$

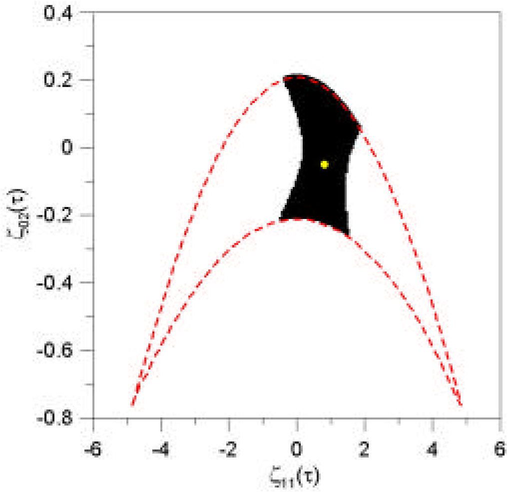
Figure 7. Erosion of the safe basin of attraction. $\Gamma_0 = 0.40$ and $\Xi_{11} = 0.10$. $\left(\dot{z}_{11} = \dot{z}_{02} = 0.0\right)$.



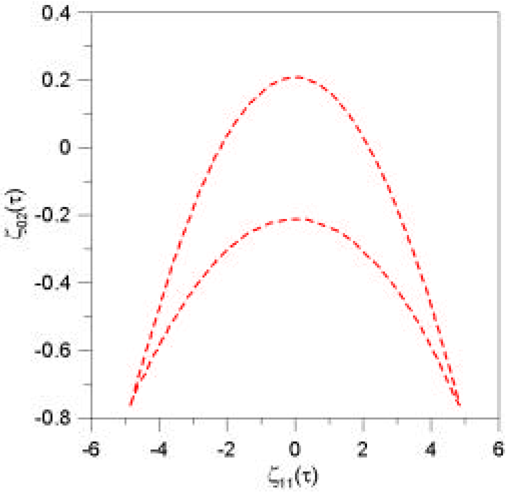
(a) $\Gamma_1 = 0.20$



(b) $\Gamma_1 = 0.25$



(c) $\Gamma_1 = 0.31$



(d) $\Gamma_1 = 0.32$

Figure 8. Erosion of the safe basin of attraction. $\Gamma_0 = 0.40$ and $\Xi_{11} = 0.50$. $\left(\dot{z}_{11} = \dot{z}_{02} = 0.0\right)$.

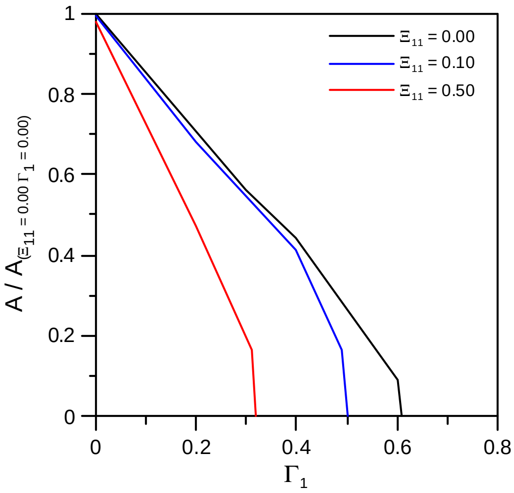


Figure 9. Variation of the normalized safe area of the basin of attraction with Γ_1, for increasing values of the imperfection magnitude Ξ_{11}. $\Gamma_0 = 0.40$.

4. CONCLUSIONS

Based on the nonlinear equations of Donnell for shallow shells, a qualitatively consistent low dimensional model is derived to study the dynamic behavior and stability of thin cylindrical shells under a suddenly applied step load of infinite duration, considering in the modeling the effects of initial geometric imperfections. The shell displacement field is described approximately by a sum of two shape functions that describes the well-known modal interaction between the critical mode and the companion axi-symmetric mode. The initial geometric imperfections are considered to have the same shape as the classical buckling or vibration mode of the shell. The influence of geometric imperfections on the value of the critical step load and on the erosion of the safe basin of attraction was studied in detail. It is concluded that the imperfections decrease the value of the critical step load, which can be much lower than the critical value for the perfect shell. The critical step load also decreases, as expected, with an increase in the static pre-load. It can also be observed that the imperfections cause an asymmetry in the basin of attraction, when compared with the basin of the perfect shell. These results can serve as a basis for the design engineers, indicating the necessary cares when they project shells submitted to sudden axial step loads. In particular, they must have in mind that geometric imperfections can appear both during the manufacture process and during the service life of the structure, causing significant reduction in its load capacity. So, proper safety factors must be used to prevent catastrophic buckling of the structure.

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