APPLICATION OF THE METHOD OF GALERKIN TO NON LINEAR PROBLEM STOCHASTIC HEAT CONDUCTION

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Abstract. *The Galerkin method is applied to the non-linear stochastic diffusion problem. The uncertainty is present in the coefficients of diffusion equation and it is modeled by random variables. The chaos polynomials is used to approximate the stochastic behavior of the problem. The approximate solutions obtained through Galerkin method are compared with Monte Carlo simulation in terms of the statistical moments of random variables generated by the random field solution.*

Keywords*: method of Galerkin, chaos polynomials, non-linear heat conduction.*

1. INTRODUCTION

In the last decades there was a significant increase of scientific works in stochastic analysis applied to the engineering systems. This fact is related to the increase of the computational capacity and the appearance of new mathematical formulations in classic numerical methods in engineering, such as the finite elements, finite differences and boundary elements.

The stochastic analysis applied to engineering systems has origin in the study of stochastic differential equations. This area consists of the existence uniqueness study; of solutions for differential equations differentiate in that the coefficients of the equation are represented by random variables or random processes. This area has been presenting a lot of interest on the part of the mathematicians for presenting a communion among several areas of knowledge of the mathematics as theory of the measure, partial differential equations, theory of the probability and etc.

The the finite elements method (FEM) has been used thoroughly in the solution of defined problems by differential equations. Some applications of FEM can be found in problems of elasticity, heat conduction, electromagnetism and others. FEM was applied in stochastic problems with the disturbance technique to treat the influence of the uncertainty in the behavior of the numerical solution. Ghanem and Spanos (1989) presented the spectral stochastic finite elements method that had a proposal different from the method of the perturbation stochastic finite elements method.

This study intends to obtain approximate solutions for the problem of heat transfer with non-linearity in the termophysics properties. Ávila and Franco (2005) presented numerical solutions for the problem of heat conduction using the method of Galerkin, for construction of the approximate solution with projection space generated by the chaos polynomials.

2. NON-LINEAR PROBLEM OF HEAT CONDUCTION

In this section, the stochastic non-linear problem of heat conduction is presented. The non-linear term appears due to dependence of the conductivity coefficient with the temperature field. This mathematical model is well-known; however, the literature presented cases where the properties are deterministic. The stochastic heat conduction in a limited interval, $\Omega \subset \mathbb{R}$, is defined as

$$\begin{cases} \nabla \bullet \left(\kappa\left(u\right) . \nabla u \right) = f, & in \quad \Omega \times \Theta \\ u = 0, & in \quad \partial\Omega \end{cases} \tag{1}$$

with $f \in L^2\left(\Omega\right)$ and

$$\kappa\left(u\right) = \kappa_0 + \kappa_1 u \ , \tag{2}$$

where κ_0, κ_1 are constants of the thermal conductivity properties. This study intends that these properties are modeled by random variables. These properties should satisfy the following requirements,

$$\exists c, C \in \mathbb{R}^+, P\left(\theta \in \Theta : \kappa_0, \kappa_1 \in \left[c, C\right]\right) = 1 \cdot \tag{3}$$

The expressed condition in Eq. (3) assures the existence and uniqueness of the stochastic non-linear problem of heat conduction.

3. NOTATION AND SPACE OF FUNCTIONS

In this section are presented some definitions and notations that will be used along this study. The principle of the causality in problems with uncertainty in the source term or in the parameters of the response of the system will present a stochastic behavior. The solution space for these problems should contain functions to present this behavior. The mathematical formulation of this study uses the approaches of the Sobolev spaces and product tensor associated to the theory of probability.

3.1. Stochastic Sobolev Spaces

The association among the theories of probability, product tensor and the Sobolev spaces of originate the stochastic Sobolev spaces. The numeric solutions obtained in these spaces, and the approach of these with the theoretical solutions is based in the isomorphism between the stochastic Sobolev spaces and Sobolev spaces of defined in more complex measure spaces, Babuška *et all* (2005) and Frauenfelder *et all* (2005). The theoretical solution is defined in the following Sobolev space

$$L^3\left(\Theta,\mathcal{F},P;H_0^1(\Omega)\right)=\left\{v:\Theta\times\Omega\to\mathbb{R}\,\middle|\,v\ \textit{is measure and}\int_\Theta\left\|v(\theta)\right\|_{H_0^1(\Omega)}^3 dP(\theta)<+\infty\right\} \quad (4)$$

The space $L^3\left(\Theta,\mathcal{F},P;H_0^1(\Omega)\right)\simeq L^3\left(\Theta,\mathcal{F},P\right)\otimes H_0^1(\Omega)$, therefore an element $u\in L^3\left(\Theta,\mathcal{F},P\right)\otimes H_0^1(\Omega)$ means that $u(x,\cdot)\in H_0^1(\Omega)$ q.s. in Θ and $u(\cdot,\theta)\in L^3\left(\Theta,\mathcal{F},P\right)$ q.s. in Ω. Let $u,v\in L^3\left(\Theta,\mathcal{F},P\right)\otimes H_0^1(\Omega)$ the following internal product is defined,

$$(u,v)=\iint_{\Omega\times\Theta}(u.v)(x,\theta)\,dxdP\,. \quad (5)$$

For the numerical solutions that will be obtained in this study the density property will be used among spaces of finite dimension generated by continuous functions. An element $u\in L^3\left(\Theta,\mathcal{F},P\right)\otimes H_0^1(\Omega)$ is defined as $(x,\theta)\mapsto\phi(x)\psi(\theta)$ with $\phi\in H_0^1(\Omega)$ and $\psi\in\bigcup_{n\in\mathbb{N}}\mathcal{P}_n(H)$ with $H\subseteq L^2\left(\Theta,\mathcal{F},P\right)$ a Gaussian Hilbert space and $\mathcal{P}_n(H)=\left\{\psi\left(\{\xi_i\}_{i=1}^M\right):\psi\ \text{is polynomials of degree}\leq n;\xi_i\in H,\forall i=1,\ldots,M;M<\infty\right\}$. The functions ψ are known as chaos polynomials and the spaces generated by these polynomials, $\bigcup_{n\in\mathbb{N}}\mathcal{P}_n(H)$, is dense in $L^2\left(\Theta,\mathcal{F},P\right)$, Wiener (1947).

4. MODELLING OF THE UNCERTAINTY

The mathematical modelling of the uncertainty is obligatory for obtaining the numeric solutions for stochastic systems. The uncertainty can be modeled through random variable or stochastic processes. In this work the modelling of the uncertainty is made through uniform random variable. They are obtained numeric solutions for two cases of modelling of the uncertainty in the coefficient of thermal conductivity κ_1. The uniform random variable has the following form

$$\varsigma(\theta)=\mu+\sigma\xi(\theta), \quad (6)$$

where ς is the property to be modeled and ξ an uniform random variable with $\langle\xi\rangle=0$ and $\langle\xi^2\rangle=1$, with that is concluded that $\mu=\langle\varsigma\rangle$ and $\sigma=\left|\sqrt{\left\langle\left(\varsigma-\langle\varsigma\rangle\right)^2\right\rangle}\right|$.

5. METHOD OF GALERKIN

The method of Galerkin will be used to obtain approximate solutions for the stochastic non-linear problem heat conduction. The space $L^3\left(\Theta,\mathcal{F},P;H_0^1(\Omega)\right)$ is the space of solution of the defined problem in Eq. (1), however the space of approximate solutions will be defined as $\mathcal{K}_n\otimes\mathcal{M}_m$ with $\mathcal{K}_n = span\left\langle\{\phi_i\}_{i=1}^n\right\rangle$ and $\phi_i\in C^1(\Omega)\bigcap C_0(\Omega)\subset H_0^1(\Omega)$ and $\mathcal{M}_m = span\left\langle\{\psi_i\}_{i=1}^m\right\rangle$ with $\overline{\mathcal{M}_m}^{L^3(\Theta,\mathcal{F},P)} = L^3(\Theta,\mathcal{F},P)$ e ψ_i's the chaos polynomials. The approximate solutions have the following form,

$$u_{nm}(x,\theta)=\sum_{i,j=1}^{n,m}u_{ij}\phi_i(x)\psi_j(\xi(\theta)) \tag{7}$$

where u_{ij} are coefficients to determine. Being put the approximate solution in the Eq. (1)

$$\varepsilon_{nm}=\sum_{i,j=1}^{n,m}\nabla\cdot\left(\kappa_0\nabla\phi_i+\kappa_1\sum_{k,l=1}^{n,m}\phi_k\nabla\phi_i\psi_l u_{kl}\right)u_{ij}\psi_j-f,\quad\forall(x,\theta)\in\Omega\times\Theta. \tag{8}$$

Eq. (8) defines the residue generated in the differential equation of the problem given in Eq. (1). It is important to point out that it was not still attributed any mathematical model, presented in Eq. (1), for the current uncertainty in the coefficient of thermal conductivity. Being taken one $\varphi\in\mathcal{K}_n\otimes\mathcal{M}_m$ defined as $\varphi(x,\theta)=\phi_p(x)\psi_q(\theta)$ and product internal as defined in Eq. (5) and imposed the minimization condition of the projection of the residue in $\mathcal{K}_n\otimes\mathcal{M}_m$

$$(\varepsilon_{nm},\varphi_{pq})=0\Rightarrow\left[\left(\kappa_0\nabla\phi_i\psi_j,\nabla\phi_p\psi_q\right)+\sum_{k,l=1}^{n,m}\left(\kappa_1\phi_k\psi_l\nabla\phi_i\psi_j,\nabla\phi_p\psi_q\right)u_{kl}\right]u_{ij}=\left(f,\nabla\phi_p\psi_q\right),\forall(p,q)\in\{1,\dots,n\}\times\{1,\dots,m\}. \tag{9}$$

Apart of the uncertainty model given to the coefficient of thermal conductivity, Eq. (9), it represents a system of non-linear equations, which can be represented as,

$$K(U)U=F, \tag{10}$$

with $K(U)=K_0+K_1(U)$. The elements of the matrix K_0 e K_1 are given by

$$K_0=\left[k_{ij}^0\right]_{n.m\times n.m}\quad\Rightarrow\quad k_{ij}^0=\left(\kappa_0\nabla\phi_i\psi_j,\nabla\phi_p\psi_q\right). \tag{11}$$

The non-linear part of the matrix K_1 is given for

$$K_1=\left[k_{ij}^1\right]_{n.m\times n.m}\quad\Rightarrow\quad k_{ij}^1=\sum_{k,l=1}^{n,m}\left[\mu_{\kappa_1}\left(\phi_k\psi_l\nabla\phi_i\psi_j,\nabla\phi_p\psi_q\right)+\sigma_{\kappa_1}\left(\xi\phi_k\nabla\phi_i\psi_l,\nabla\phi_p\psi_q\right)\right]u_{kl}. \tag{12}$$

One can observe that in both cases the current non-linear terms in the matrix K_1 are polynomial type. To obtain the solution of Eq. (10) the Newton-Raphson method will be used. For such the residue $R:\mathbb{R}^{n.m}\to\mathbb{R}^{n.m}$ following

$$R(U)=K(U)U-F. \tag{13}$$

The solution of the non-linear system in Eq. (10) will occur when $R(U^*)=0$ with $U^*\in\mathbb{R}^{n.m}$, assumed that the residue function is expanded analytically in Taylor series of first order in $U\in\mathbb{R}^{n.m}$,

$$R^1(U^*)=R(U)+\nabla_U R(U)(U^*-U). \tag{14}$$

Impose the condition $R^1(U^*)=0$ in Eq. (14), one can be determined U^* as

$$U^* = U - \left[K_0 + K_1(U) + \nabla_U K_1(U)U\right]^{-1}\left[K(U)U - F\right]. \tag{15}$$

The Eq. (15) is a recursive equation to determine the solution of the non-linear system defined in Eq. (10). For a given precision for the system solution, Eq. (10), the iterative process will be concluded when one verifies the following conditions,

$$\left\|U^* - U\right\| < \varepsilon_U \quad \wedge \quad \left\|R\left(U^*\right)\right\| < \varepsilon_R, \tag{16}$$

where $\varepsilon_U, \varepsilon_R \in \mathbb{R}^+$, are arbitrary tolerances. The expressed conditions in Eq. (16) constitute the finalize criterion of the iterative process used in this study.

6. NUMERIC RESULTS

In this section the numeric solutions are presented for the unidimensional problem non linear stochastic heat conduction. A numeric example is presented in that the uncertainty on the property is modeled by a uniform random variable,

$$\kappa_1(\theta) = \mu_{\kappa_1} + \sigma_{\kappa_1}\xi(\theta), \tag{17}$$

where $\mu_{\kappa_1} = 1000$ W.m^2 and $\sigma_{\kappa_1} = 10$ W.m^2. The property $\kappa_0 = 1000$ W.m is considered as deterministic. The domain is defined for $\Omega = (0, L)$ with $L = 1$ m and it is submitted a generation of heat $q(x) = 1000$ W/m, $\forall x \in \Omega$. The statistical moments of first and second order of the random field of temperatures are the parameters of evaluation of the numeric solutions obtained by the Galerkin method. The Monte Carlo simulation is used to accomplish the evaluation of the Galerkin method. The statistical moments was obtained from 1000 realizations of the non-linear stochastic problem of heat conduction, $N_S = 1000$. To quantify the approximation between the statistical moments of first and second order is defined the error average value function $\left(E_{\mu_u}\right)$ and the variance error function $\left(E_{V_u}\right)$, respectively. The error average value function, $E_{\mu_u} : \Omega \to \mathbb{R}^+$, is defined as,

$$E_{\mu_u}(x) = \left|\left(\widehat{\mu}_u - \mu_u\right)(x)\right|, \tag{18}$$

where $\widehat{\mu}_u$ is the average value obtained through Monte Carlo and μ_u the average value through the method of Galerkin. The variance error function, $E_{V_u} : \Omega \to \mathbb{R}^+$, is defined as,

$$E_{V_u}(x) = \left|\left(\widehat{V}_u - V_u\right)(x)\right|, \tag{19}$$

where $\widehat{V}_u$ is variance function obtained through Monte Carlo and V_u is the variance function obtained by means of Galerkin method. In Fig. 1a and 1b are shown the graphs of the average value functions and error in average value, respectively, obtained through simulation Monte Carlo simulation and for the Galerkin method.

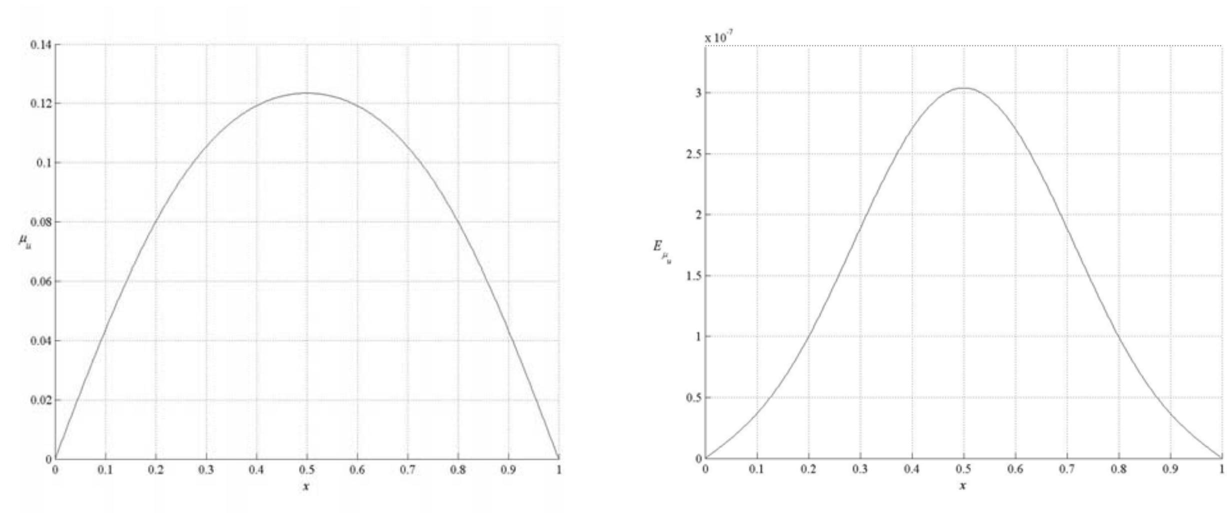

Figure 1: a) Average value of temperature field; b) Error in average value.

Fig. 1b shows a good approximation between the average value function obtained through the Galerkin method and that obtained through Monte Carlo. It is important to mention that such approach was with only two chaos polynomials. In Fig. 2a and 2b are shown the graphs of the functions variance and error in variance, respectively, obtained through simulation of Monte Carlo and for the method of Galerkin.

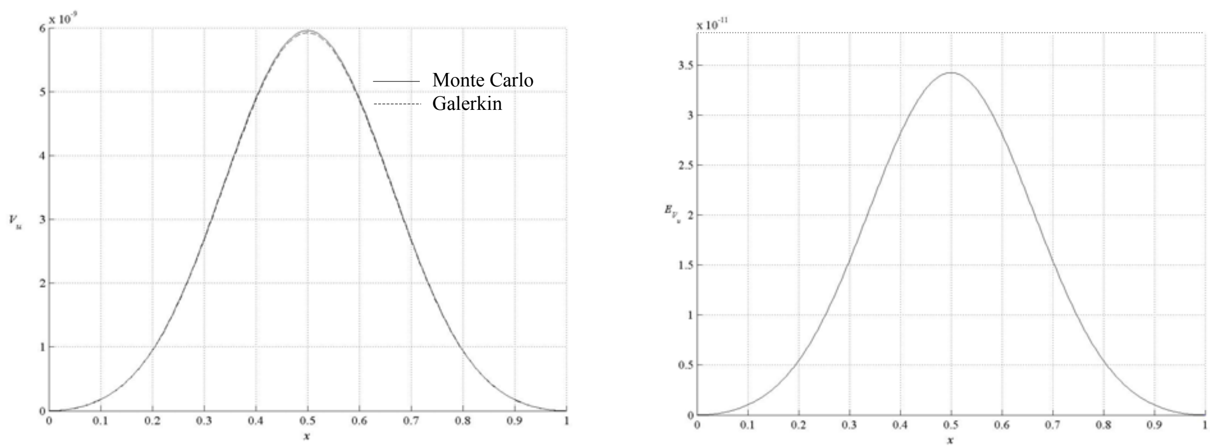

Figure 2: a) Variance function temperature field; b) Error in variance.

The Fig. 2b shows a loss in the quality of the approach of the function variance, relatively the approach that was obtained for the average value function of the random field of temperatures.

7. CONCLUSION

In this work the method of Galerkin was applied to obtain numeric solutions to the problem non linear stochastic heat conduction. The uncertainty on the coefficient of thermal conductivity was modeled by a uniform random variable. The statistical moments of first and second order obtained from the numeric solutions showed a satisfactory approach. It was observed that the quality of the approach decreases for the second statistical moment.

8. REFERENCES

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