APPLICATION OF THE GALERKIN METHOD TO THE PROBLEM OF BENDING OF PLATES IN RANDOM ELASTIC FOUNDATION

Cláudio R. Ávila da S. Jr., avila@utfpr.edu.br
Federal University of Technology - Parana, September Seven Avenue, 3165, Curitiba, Parana, Brazil.

Abstract. *In this study the model of Winkler is applied to the problem of bending of plates in a random foundation elastic. The uncertainty on the Winkler module is modeled by random variables. The Galerkin method with projection on the space generated by chaos polynomials is used to obtain approximate solutions. The uncertainty is discretized through the chaos polynomials. The statistics obtained by the Galerkin method are compared with those obtained by Monte Carlo simulation.*

Keywords: *Winkler foundation, bending of plates*, *chaos polynomials, Galerkin method.*

1. INTRODUCTION

The modelling and analysis of the structural behavior of a mechanical system are the most important steps in the conventional methodology of mechanical project. In this activity, the simulation of the structural acting for the project and the structural behavior is very important and it has been growing in the last decades, due to the appearance of new methods of analysis and growth of the capacity computational. Because of that, the approximate methods, as the finite elements method (FEM), and finite differences are had diffused inside of the analysis area. In this line the research of robust methods can be observed and perfected for the solution of problems with a higher complexity degree. These problems are considered as new physical effects, incorporating to the mathematical model a non-linear component of the global behavior of the system. These progresses have the purpose of obtaining a mathematical model to get, or be close the, to describe the real behavior of the analyzed system.

The increase of the complexity in the description of the constitutive models, the consistence and robustness of the mathematical models are not enough to model the behavior of the randomness of the problem. For a simple fact, you don't consider them in the modelling of the problem. For instance, the behavior of a metallic plate submitted to bending, cannot be satisfactorily described if the geometric imperfections of random nature was not considered, that can be originating from the lamination process. Besides, the influence of the randomness can be intensified in function of the evaluation type which the system will be submitted. In this way, the treatment and modelling of the randomness of the system constitute a new subject of scientific investigation.

In this study, the Galerkin method is applied to obtain approximate solutions, in stochastic Sobolev spaces (Matthies and Keese, 2005), for the problem of bending of plates in Winkler's foundation of with uncertainty in the Young's modulus. The cases are examined in that the uncertainty can be present in the Young modulus or in the stiffness of the foundation. The uncertainty on the mechanical properties will be modeled through uniform random variables. The Askey-Wiener (Xiu and Karniadakis, 2002) is used to represent the uncertainty through the chaos polynomials.

2. PROBLEM OF BENDING OF PLATES IN WINKLER FOUNDATION

The problem of bending of Kirchoff's plates in limited domain is modeled by

$$\begin{aligned} &\Delta\left(D\Delta u\right)+\alpha u=f, && \text{in} \quad \Omega \\ &u=0, && \text{in} \quad \partial\Omega_u \\ &\frac{\partial u}{\partial n}=0, && \text{in} \quad \partial\Omega_{\partial u} \end{aligned} \tag{1}$$

where u is the transverse displacement of the plate, D the bending rigidity module of the plate and the stiffness foundation. Numeric solutions for two cases of the uncertainty will be obtained. In the former, the uncertainty is present in the stiffness bending of the plate, and latter, the uncertainty is on the stiffness foundation. However, in both cases the uniform random variable has the following form

$$\varsigma\left(\theta\right)=\mu+\sigma\xi\left(\theta\right), \tag{2}$$

where ς is the mechanical property to be modeled and ξ a uniform random variable with and $\langle \xi \rangle = 0$ and $\langle \xi^2 \rangle = 1$, concluding that $\mu = \langle \varsigma \rangle$ and $\sigma = \left| \sqrt{\left\langle \left(\varsigma - \langle \varsigma \rangle \right)^2 \right\rangle} \right|$.

3. STOCHASTIC SOBOLEV SPACES

The association among the theories of probability, product space and the Sobolev spaces of originate the stochastic Sobolev spaces. The numeric solutions obtained in these spaces, and the approach of these with the theoretical solutions is based on the isomorphism between the stochastic Sobolev spaces and Sobolev spaces defined in more complex spaces of measurement (Babuška *et all,* 2005, and Frauenfelder *et all,* 2005). The theoretical solution is defined in the following Sobolev space

$$L^2\left(\Theta, \mathcal{F}, P; H_0^2(\Omega)\right) = \left\{ v : \Theta \times \Omega \to \mathbb{R} \middle| v \ \textit{is measure and} \int_{\Theta} \| v(\theta) \|^2_{H_0^2(\Omega)} dP(\theta) < +\infty \right\} \tag{3}$$

The space $L^2\left(\Theta, \mathcal{F}, P; H_0^2(\Omega)\right)$ is isomorphic to $L^2\left(\Theta, \mathcal{F}, P\right) \otimes H_0^2(\Omega)$, therefore an element $u \in L^2\left(\Theta, \mathcal{F}, P\right) \otimes H_0^2(\Omega)$ means that $u(x,\cdot) \in H_0^2(\Omega)$ q.s. in Θ and $u(\cdot,\theta) \in L^2\left(\Theta, \mathcal{F}, P\right)$ a.e. in Ω. For $u, v \in L^2\left(\Theta, \mathcal{F}, P\right) \otimes H_0^2(\Omega)$, the following internal product is defined,

$$(u,v) = \iint_{\Omega \times \Theta} (u.v)(x,\theta)\, dxdP\,. \tag{4}$$

For the numeric solutions that will be obtained here, the density property will be used among spaces of finite dimension generated by continuous functions. An element $u \in L^2\left(\Theta, \mathcal{F}, P\right) \otimes H_0^2(\Omega)$ is defined as $(x,\theta) \mapsto \phi(x)\psi(\theta)$ with $\phi \in H_0^2(\Omega)$ and $\psi \in \bigcup_{n \in \mathbb{N}} \mathcal{P}_n(H)$ with $H \subseteq L^2\left(\Theta, \mathcal{F}, P\right)$ a Gaussian Hilbert space and $\mathcal{P}_n(H) = \left\{ \psi\left(\{\xi_i\}_{i=1}^M \right) : \psi \text{ is polynomials of degree} \le n; \xi_i \in H, \forall i = 1, \dots, M; M < \infty \right\}$. The functions ψ are known as polynomials of Chaos and the spaces generated by these polynomials, $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n(H)$, is dense in $L^2\left(\Theta, \mathcal{F}, P\right)$, Wiener (1947).

4. GALERKIN METHOD

The Galerkin method will be used to obtain approximate solutions for the bending of plates in Winkler's foundation. The space $L^2\left(\Theta, \mathcal{F}, P; H_0^2(\Omega)\right)$ is that of solution defined in Eq. (1). However, the space of approximate solutions will be defined as $\mathcal{K}_n \otimes \mathcal{M}_m$ and $\mathcal{K}_n = span\left\langle \{\phi_i\}_{i=1}^n \right\rangle$ with $\phi_i \in C^2(\Omega) \cap C_0^1(\Omega) \subset H_0^2(\Omega)$ and $\mathcal{M}_m = span\left\langle \{\psi_i\}_{i=1}^m \right\rangle$ with $\overline{\mathcal{M}_m}^{L^2(\Theta,\mathcal{F},P)} = L^2\left(\Theta, \mathcal{F}, P\right)$ and ψ_i's the chaos polynomials. The approximate solutions have the following form,

$$u_{nm}(x,\theta) = \sum_{i,j=1}^{n,m} u_{ij} \phi_i(x) \psi_j(\xi(\theta)) \tag{5}$$

where u_{ij} are coefficients to solve. The approximate solution in Eq. (1)

$$\varepsilon_{nm} = \sum_{i,j=1}^{n,m} \left[\Delta(D\Delta\phi_i) + \alpha\phi_i \right] u_{ij}\psi_j - f, \quad \forall (x,\theta) \in \Omega \times \Theta. \tag{6}$$

Eq. (6) defines the residue generated in the differential equation of the problem given in Eq. (1). It is important to point out that the current uncertainty in the coefficient of elasticity was not attributed, in the mathematical model, presented in Eq. (6). Being $\varphi \in \mathcal{K}_n \otimes \mathcal{M}_m$ defined as $\varphi(x,\theta) = \phi_p(x)\psi_q(\theta)$, the internal product defined in Eq. (4), and imposing the minimization condition of the projection of the residue in $\mathcal{K}_n \otimes \mathcal{M}_m$

$$\left(\varepsilon_{nm},\varphi_{pq}\right)=0 \Rightarrow \sum_{i,j=1}^{n,m}\left[\left(D\Delta\phi_i\psi_j,\Delta\phi_p\psi_q\right)+\left(\alpha\phi_i\psi_j,\phi_p\psi_q\right)\right]u_{ij}=\left(f,\phi_p\psi_q\right),\ \forall(p,q)\in\{1,...,n\}\times\{1,...,m\}. \tag{7}$$

Eq. (7) represents a system of linear equations,

$$KU = F. \tag{8}$$

The elements of the matrix will change as the uncertainty on the coefficients of the mechanical properties is attributed. For the case in that the uncertainty is on the stiffness bending

$$K=\left[k_{ij}\right]_{n.m\times n.m} \quad \Rightarrow \quad k_{ij}=\mu_D\left(\Delta\phi_i\psi_j,\Delta\phi_p\psi_q\right)+\sigma_D\left(\xi\Delta\phi_i\psi_j,\Delta\phi_p\psi_q\right)+\alpha\left(\phi_i\psi_j,\phi_p\psi_q\right). \tag{9}$$

For the case in that the uncertainty is on the coefficient α

$$K=\left[k_{ij}\right]_{n.m\times n.m} \quad \Rightarrow \quad k_{ij}=D\left(\Delta\phi_i\psi_j,\Delta\phi_p\psi_q\right)+\mu_\alpha\left(\phi_i\psi_j,\phi_p\psi_q\right)+\sigma_\alpha\left(\xi\phi_i\psi_j,\phi_p\psi_q\right). \tag{10}$$

This study will not treat of problems where the uncertainty is present in the loading term.

5. NUMERIC RESULTS

In this section, the numerical solutions are presented for the bending of the plates in a Winkler's foundation. Two numerical examples and the uncertainty are presented on the mechanical properties is modeled by a random variable of the uniform type. The domain plate is $\Omega = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, 0<x<L_x,\ 0<y<L_y\right\}$ with $L_x = L_y = 1\,\text{m}$ and it is submitted to a distributed load $q(x,y) = 1000\ \text{N/m}^2$, $\forall(x,y)\in\Omega$. In both examples the expected values of the stiffness bending and the foundation are given by $\mu_D = 1000$ N.m and $\mu_\alpha = 100\ \text{N/m}^3$, respectively. The statistical moments of first and second order of the random field of transverse displacement are the parameters of evaluation of the numeric solutions obtained by the Galerkin method. The Monte Carlo simulation is used to accomplish the evaluation of the Galerkin method. To quantify the approximation between the statistical moments of first and second order is defined the average value error function $\left(E_{\mu_u}\right)$ and the variance error function $\left(E_{V_u}\right)$, respectively. The average value error function, $E_{\mu_u}:\Omega\to\mathbb{R}^+$, is defined as,

$$E_{\mu_u}(x,y)=\left|\left(\widehat{\mu_u}-\mu_u\right)(x,y)\right|, \tag{11}$$

where $\widehat{\mu_u}$ is the average value obtained through Monte Carlo and μ_u the average value through the Galerkin method. The variance error function, $E_{V_u}:\Omega\to\mathbb{R}^+$, is defined as,

$$E_{V_u}(x,y)=\left|\left(\widehat{V_u}-V_u\right)(x,y)\right|, \tag{12}$$

where $\widehat{V_u}$ is the variance function obtained through Monte Carlo simulation and V_u the variance function obtained through the Galerkin method.

5.1 Example 1

In this example is assumed that the uncertainty is present in the stiffness bending of the plate, considering as deterministic the stiffness foundation. The random variable that models the uncertainty on the stiffness is defined as,

$$D(\theta) = \mu_D + \sigma_D \xi(\theta), \tag{13}$$

with $\sigma_D = 100$ N.m and ξ an uniform random variable. Fig. 1a and 1b show the graphs of the average value functions obtained through Monte Carlo simulation and by the Galerkin method, respectively.

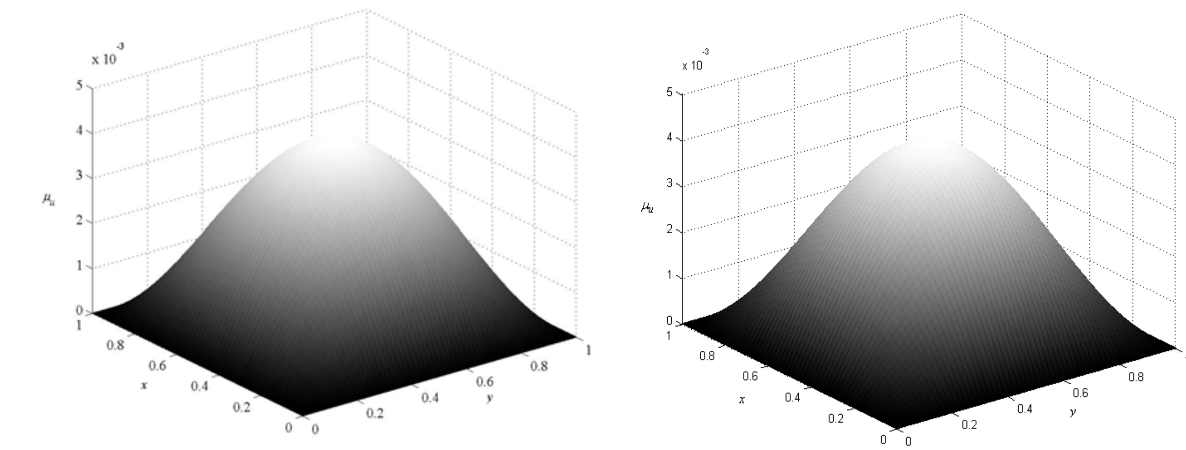

Figure 1: a) Average value obtained by Galerkin; b) Average value obtained by Monte Carlo.

The comparison between Fig. 1a and 1b aloows concluding that there is a satisfactory approximation between the average value functions obtained by Monte Carlo and Galerkin. Fig. 2a and 2b present the graphs of the variance functions obtained through Monte Carlo simulation and by the Galerkin method, respectively. A good approximation is observed between the variance functions obtained through both methodologies.

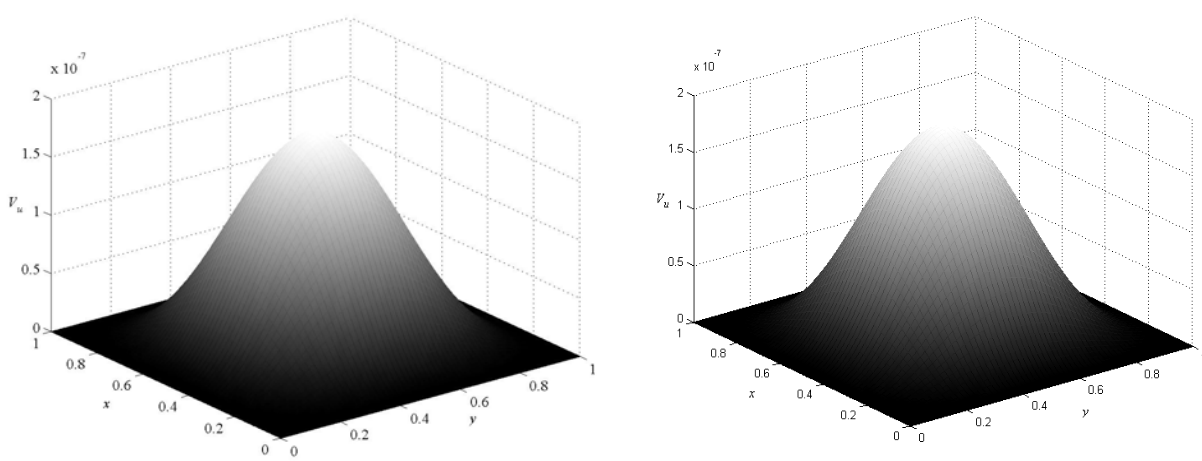

Figure 2: a) Variance function obtained by Galerkin; b) Variance function obtained by Monte Carlo.

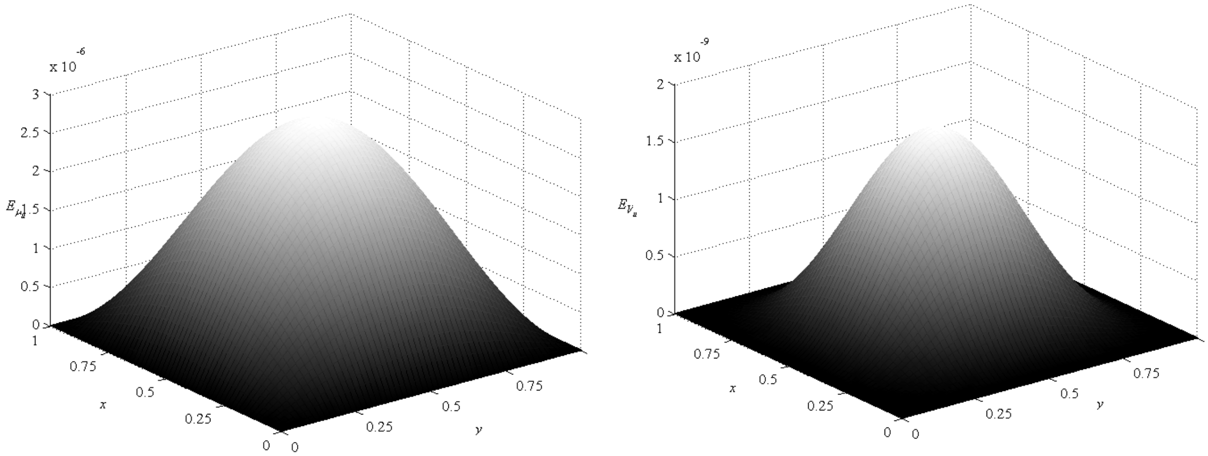

Figure 3: a) Average value error function; b) Variance error function.

Figs. 3a and 3b present the graphs of the average value error and variance error function, respectively. Comparing all graphs of Figs. 2 and 3, relatively, one concludes that the average value error function reached a smaller value than the variance error function.

5.2 Example 2

In this example it is assumed that the uncertainty is present in the stiffness foundation, considering the stiffness as deterministic. The random variable that models the uncertainty on the stiffness foundation is defined as,

$$\alpha(\theta) = \mu_\alpha + \sigma_\alpha \xi(\theta), \tag{14}$$

with $\sigma_\alpha = 10 \text{ N/m}^3$ and ξ as an uniform random variable. Figs. 4a and 4b present the graphs of the average value functions obtained through Monte Carlo simulation and by the Galerkin method, respectively.

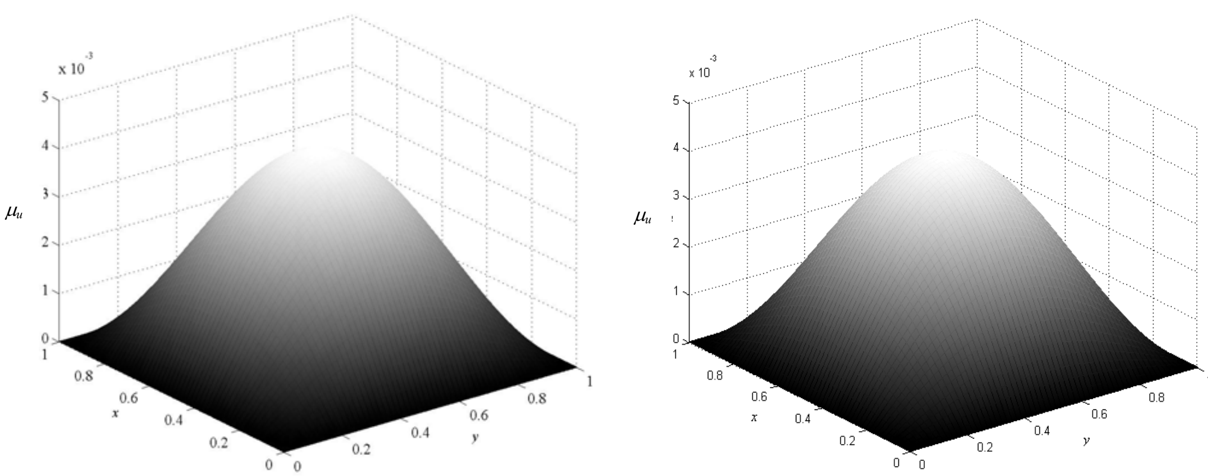

Figura 4: a) Average value obtained by Galerkin; b) Average value obtained by Monte Carlo.

Similarly to the observed in Example 1, through the comparison between Fig. 4a and 4b, a satisfactory approximation between the average value functions obtained by Monte Carlo simulation and Galerkin method is observed. It is important to point out that the graphs presented in Fig. 4 are similar to the graphs of Fig. 1. This similarity is due to the fact of the average value function only depends on the average values of the elastic properties.

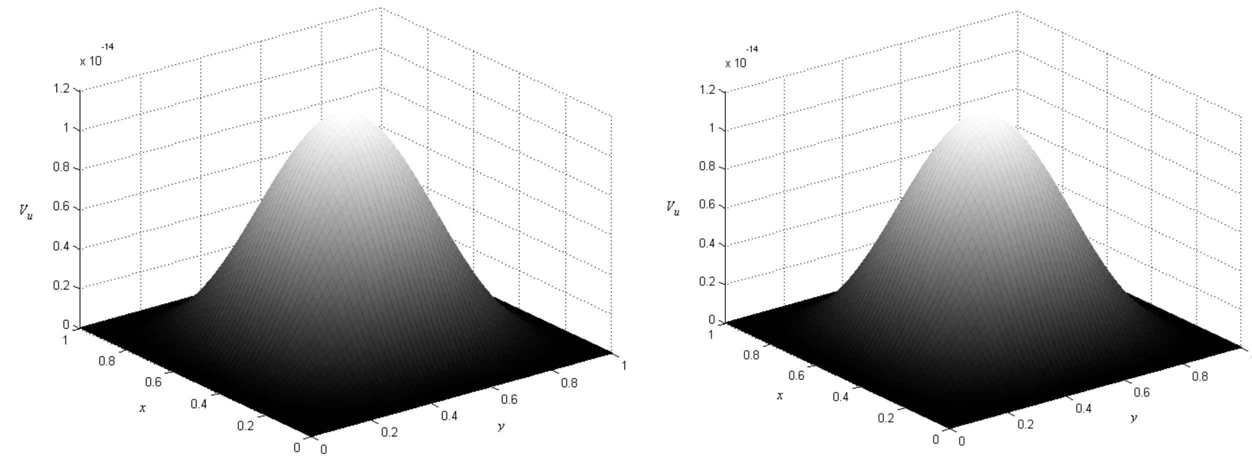

Figure 5: a) Variance function obtained by Galerkin; b) Variance function obtained by Monte Carlo.

Figs. 5a and 5b present the graphs of the variance functions obtained through Monte Carlo simulation and by the Galerkin method, respectively. A good approximation is observed between the variance functions obtained by both methodologies.

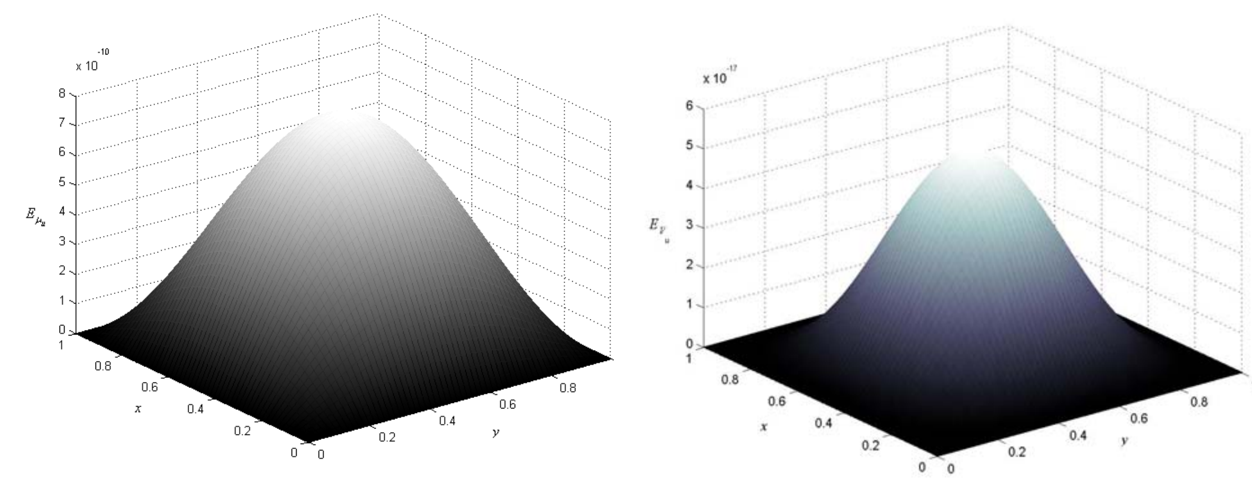

Figure 6: a) Average value error function; b) Variance error function.

Figs. 6a and 6b present the graphs of the average value and variance error functions, respectively. Comparing the Figs. 5 and 6, relatively, one can observe that the average value error function reaches a smaller value than the variance error function.

6. CONCLUSION

In this study, the numeric solutions for the problem of bending of plates in foundation of Winkler with the current uncertainty in the elastic coefficients of the plate and the foundation were presented. The uncertainty on the coefficients was modeled through uniform random variables. The Galerkin method presented satisfactory results in the approximation, from the numeric solution, of the first and second statistical moments of the random field of transverse displacement. Specifically, the case that the uncertainty was present in the stiffness foundation, presented better results. The error function in statistical moments showed that the approximations lost quality as it increased the statistical moments degree.

7. REFERENCES

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8. RESPONSIBILITY NOTICE

The author is the only responsible for the printed material included in this paper.

9. ACKNOWLEDGMENT

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