A COMPARATIVE STUDY OF THE TWO BOUNDARY ELEMENT APPROACHES BASED ON THE CONVOLUTION QUADRATURE METHOD

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Abstract. *In this work two time domain Boundary Element formulations are discussed. The main feature of the developed approaches is the fact that both are based on the so-called Convolution Quadrature Method (CQM). In the CQM formulation, the convolution integral in the boundary integral equations is numerically approximated by a quadrature formula whose weights depend only on the Laplace transformed fundamental solution and a linear multistep method. One of the presented approaches uses the boundary element method (BEM) based on CQM (CQM-BEM) to solve elasticity problems. The other approach uses the CQM-BEM for the analysis of scalar wave propagation problems.*
In the early stages of the numerical implementation, in spite of the good results obtained, the CQM-BEM approach turned out to be not an ideal formulation, when compared with other existing alternatives, due its computational cost. Only when a technique similar to the Fast Fourier Transform (FFT) was introduced this difficulty was overcome producing acceptable CPU time in comparison with the previous existing formulations. The advance of this last topic is an important point of discussion in this work and is demonstrated with numerical results. Also, a comparison involving two-dimensional problems is included to illustrate the two presented formulations.

Keywords*: Boundary Element Method, Convolution Quadrature Method, scalar wave equation, elasticity.*

1. INTRODUCTION

In recent years the authors have been working with the Boundary Element Method (BEM) and the Convolution Quadrature Method (CQM) to solve 2D elastodynamic problems. Some publications related to this topic (Vera-Tudela & Telles, 2003 & 2005, Abreu, Carrer & Mansur, 2001 & 2003) have appeared and studies have been devoted to solve critical problems such as the excessive computational time during processing.

The BEM (Brebbia, Telles and Wrobel, 1984) transforms the partial differential equation that describes an engineering or scientific problem by means of unknown variables inside and on the boundary of the domain into an integral equation involving only boundary values and then finds the numerical solution of this equation. Since all numerical approximations take place only on the boundaries, the dimensionality of the problem is reduced by one and smaller system of equations are obtained in comparison with those achieved through differential methods.

Elastodynamics is one of the most important topics studied with the BEM. In general, the treatment of 2D elastodynamic problems with the BEM can be summarized in three main approaches: one is a direct formulation in the time domain (Mansur, 1983) that is solved evaluating the classic BEM formulation in conjunction with conventional step-by-step time integration schemes. The second is the analysis in a transform domain using the Laplace transformation (Cruse, 1968). In this case, a numerical inverse transformation was required to bring the transformed solution back to the original time domain. The third is the dual reciprocity method where one of the advantages is that dynamic problems can be solved by a static fundamental solution (Partridge, Brebbia & Wrobel, 1992). Recently, an alternative procedure, so-called the Convolution Quadrature Method (CQM) (Lubich, 1988, 1988a) has emerged as an elegant way of dealing with the inverse transformation procedure. Here, in a time-dependent BEM formulation, the convolution integral is substituted by a quadrature formula, whose weights are computed using the Laplace transform of the fundamental solution and a linear multistep method. In spite of this, the final solution is directly obtained in the time domain.

In the study of 2D elastodynamic and scalar problems solved by the TD-BEM and CQM, it was observed that the simulation incurs in a significant computational cost. This cost is fundamentally due to the computation of a temporal convolution over N time steps requiring $O(N^2)$ operations and $O(N)$ memory per expansion coefficient. In the present work, a Fast Fourier Transform technique is implemented in the calculus of the quadrature weights of the CQM formulation in order to reduce the operations to $O(N\log N)$.

An example is presented to validate the implementation allowing for the reduction in computational cost of the CQM-BEM implementation.

2. THE CONVOLUTION QUADRATURE METHOD

The Convolution Quadrature Method (CQM) developed by Lubich, (1988, 1988a) numerically approximates a convolution integral of a function $y(t)$ of $u(t-\tau)$ and $f(t)$ by means of discrete values of $\hat{u}(s)$ and $f(t)$, where $\hat{u}(s)$ is the Laplace transform of $u(t)$. The convolution integral is written as:

$$y(t) = \int_0^t u(t-\tau)\ f(\tau)\ d\tau = u(t) * f(t) \tag{1}$$

In this way, Eq. (1) can be written in a discretized form at points $n\Delta t$ as:

$$y\ (n\Delta t) = \sum_{k=0}^{n} \omega_{n-k}(\Delta t)\ f(k\Delta t)\ ,\ n = 0, 1, \ldots,\ N \tag{2}$$

where Δt is the time interval sampling and N is the total number of time sampling.

In Eq. (2), ω_n are the integration weights which represent the coefficients of a power series of complex variable z; this series approximates the Laplace transform $\hat{u}(s)$ as follows:

$$\hat{u}(s) = \hat{u}\left(\frac{\gamma(z)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n\ \Delta t\ z^n \tag{3}$$

where:

$$\omega_n(\Delta t) = \frac{1}{2\pi i} \int_{|z|=\rho_r} \hat{u}\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1} dz \approx \frac{\rho_r^{-n}}{L} \sum_{l=0}^{L-1} \hat{u}\left(\frac{\gamma\left(\rho_r e^{i\, l\, 2\pi / L}\right)}{\Delta t}\right) e^{-i\, n\, l\, 2\pi / L} \tag{4}$$

where ρ_r is the radius of a circle in the domain of analyticity of the function $\hat{u}(s)$. In Eq. (4) a polar coordinate system was adopted and the integral was approximated by a trapezoidal rule with L equals steps ($2\pi / L$).

The function $\gamma(z)$, previously utilized in Eqs (3) and (4), is the quotient of the polynomials generated by a linear multistep method. If an error δ is assumed in the computation of $\hat{u}(s)$ present in Eq. (4), the choice of $L = N$ and $\rho^N = \sqrt{\delta}$ leads to an error in ω_n of order $O\left(\sqrt{\delta}\right)$. For more details concerning the CQM references Lubich, 1988, 1988a and 1994 are indicated.

The CQM procedure can be applied in the numerical resolution of a time-domain BEM formulation when the Laplace transformed of the fundamental solution is known. In the classical time-domain BEM formulation the matrices ***H*** and ***G*** are generated and for all boundary nodes a complete system of equations is formed as follows:

$$\boldsymbol{C}\,\boldsymbol{u}^n = \sum_{k=0}^{n} \boldsymbol{G}^{n-k}\boldsymbol{p}^k - \sum_{k=0}^{n} \boldsymbol{H}^{n-k}\boldsymbol{u}^k \tag{5}$$

Here, ***C*** is a quasi diagonal matrix that is formed by the coefficients $C_{ij}(\xi)$; n and k correspond to the variables of the time discretization $t_n = n\Delta t$ and $t_k = k\Delta t$, respectively.

After imposing the boundary conditions, Eq. (5) can be written as:

$$\boldsymbol{A}^0\boldsymbol{y}^n = \boldsymbol{f}^n + \sum_{k=0}^{n-1} \boldsymbol{f}^k \tag{6}$$

where $\boldsymbol{A}^0$ is the matrix of known coefficients that contains the contribution of first time only, i.e., at $t=0$. Vector $\boldsymbol{y}^n$ is the final response on the time domain for each time $t_n = n\Delta t$, i.e., unknowns and contribution of the boundary conditions at time t_n are stored, respectively, in vectors $\boldsymbol{y}^n$ and $\boldsymbol{f}^n$. Vector $\boldsymbol{f}^k$ contains the contribution of previous time and it is given by:

$$\boldsymbol{f}^k = \boldsymbol{G}^{n-k}\boldsymbol{p}^k - \boldsymbol{H}^{n-k}\boldsymbol{u}^k \quad (7)$$

3. THE CQM APPLIED TO THE BEM: ELASTICITY APPROACH

Consider an elastic solid enclosed by a boundary Γ, subjected to specified external dynamic loadings and in the absence of body forces. The condition of dynamic equilibrium of the body is expressed by the equation:

$$\mu\, u_{i,jj} + (\lambda+\mu)\, u_{j,ji} = \rho\, \ddot{u}_i \quad (8)$$

In the above equation μ and λ are the Lame's constants: $\mu = \frac{E}{2(1+\nu)}$ and $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$, E is the Young´s modulus, ν is the Poisson´s ratio, ρ is the mass density and $\ddot{u}_i$ are acceleration components. To uniquely formulate the dynamic problem, boundary and initial conditions, which specify the state of displacements and velocities at time t_0, must be imposed. Following the usual procedure of the BEM formulation, the starting integral equation can be expressed as follow:

$$4\pi C_{ij}(\xi)\, u_j(\xi,t) = \int_0^{t^+}\int_\Gamma u^*_{ij}(x,t;\xi,\tau)\, p_j(x,\tau)\, d\Gamma\, d\tau - \int_0^{t^+}\int_\Gamma p^*_{ij}(x,t;\xi,\tau)\, u_j(x,\tau)\, d\Gamma\, d\tau \quad (9)$$

where u_j and p_j are boundary displacements and tractions, respectively; u^*_{ij} is the fundamental solution; p^*_{ij} is the fundamental traction and C_{ij} is the usual free coefficient dependent on the location of ξ (interior or boundary).

Following the CQM procedure, the convolution integrals presented in Eq. (9) can be approximated by:

$$\int_0^{t^+} u^*_{ij}(x,t;\xi,\tau)\, p_j(x,\tau)\, d\tau = \sum_{k=0}^{n} {}^{n-k}\boldsymbol{g}^e_{ij}(x,\xi,\Delta t)\; {}^k p^e_j(x) \qquad n = 0,1,\ldots,N \quad (10)$$

and

$$\int_0^{t^+} p^*_{ij}(x,t;\xi,\tau)\, u_j(x,\tau)\, d\tau = \sum_{k=0}^{n} {}^{n-k}\boldsymbol{h}^e_{ij}(x,\xi,\Delta t)\; {}^k u^e_j(x) \qquad n = 0,1,\ldots,N \quad (11)$$

The weights $\boldsymbol{g}$ and $\boldsymbol{h}$ in Eqs. (10) and (11) are computed, respectively, with the expressions:

$${}^n\boldsymbol{g}^e_{ij}(x,\xi,\Delta t) = \frac{\rho_r^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma^e}\hat{u}^*_{ij}\left(x,\xi,\frac{\gamma\left(\rho_r\, e^{i\, l\, 2\pi/L}\right)}{\Delta t}\right)\Phi^e(x)\, d\Gamma\, e^{-i\, n\, l\, 2\pi/L} \quad (12)$$

and

$${}^n\boldsymbol{h}^e_{ij}(x,\xi,\Delta t) = \frac{\rho_r^{-n}}{L}\sum_{l=0}^{L-1}\int_{\Gamma^e}\hat{p}^*_{ij}\left(x,\xi,\frac{\gamma\left(\rho_r\, e^{i\, l\, 2\pi/L}\right)}{\Delta t}\right)\Phi^e(x)\, d\Gamma\, e^{-i\, n\, l\, 2\pi/L} \quad (13)$$

where $e = 1, 2, \ldots, E_{le}$, E_{le} is the number of elements, $\Phi^e(x)$ represents the interpolation function utilized in the boundary discretization, $\hat{u}^*_{ij}$ and $\hat{p}^*_{ij}$ are the Laplace transform of the fundamental solution and the Laplace transform of the fundamental traction, respectively, that for an elastodynamic 2D problem (Barra, 1996) are given by:

$$\hat{u}^*_{ij}(x,\xi,s) = \frac{1}{\rho c_s^2}\left[\varphi(r)\delta_{ij} - \chi(r) r,_i r,_j\right] \tag{14}$$

and

$$\hat{p}^*_{ij}(x,\xi,s) = \left\{\left[\frac{d\varphi(r)}{dr} - \frac{\chi(r)}{r}\right]\left(\delta_{ij}\frac{\partial r}{\partial n} + r,_j n_i\right) - 2\frac{\chi(r)}{r}\left(n_j r,_i - 2 r,_i r,_j \frac{\partial r}{\partial n}\right)\right.$$
$$\left. -2\frac{d\chi(r)}{dr} r,_i r,_j \frac{\partial r}{\partial n} + \left(\frac{c_p^2}{c_s^2} - 2\right)\left(\frac{d\varphi(r)}{dr} - \frac{d\chi(r)}{dr} - \frac{\chi(r)}{r}\right) r,_i n,_j\right\} \tag{15}$$

where s is the parameter of the Laplace transform and the functions $\chi(r)$ and $\varphi(r)$ are defined as follow:

$$\chi(r) = K_2\left(\frac{sr}{c_s}\right) - \frac{c_s^2}{c_p^2} K_2\left(\frac{sr}{c_p}\right) \tag{16}$$

and

$$\varphi(r) = K_0\left(\frac{sr}{c_s}\right) + \left(\frac{sr}{c_s}\right)^{-1}\left[K_1\left(\frac{sr}{c_s}\right) - \frac{c_s}{c_p} K_1\left(\frac{sr}{c_p}\right)\right] \tag{17}$$

In the above equations r is the distance between ξ and x; c_p is the P-wave velocity and c_s is the S-wave velocity; K_j is the modified Bessel function of the second kind and δ_{ij} is the Kronecker delta.

Following the CQM procedure Eq. (9) is rewritten in a discretized form as:

$$4\pi C_{ij}(\xi) u_j(\xi, t_n) = \sum_{e=1}^{E_{le}} \sum_{k=0}^{n} {}^{n-k}\boldsymbol{g}^e_{ij}(x,\xi,\Delta t)\, {}^{k}p^e_j(x) - \sum_{e=1}^{E_{le}} \sum_{k=0}^{n} {}^{n-k}\boldsymbol{h}^e_{ij}(x,\xi,\Delta t)\, {}^{k}u^e_j(x) \tag{18}$$

Equation (18) can now be written for all boundary nodes in terms of global matrices to give the complete system of equations having a similar form of Eq. (5). Finally, the procedure follows the same steps previously presented in section **2** of the Convolution Quadrature Method.

In the present work, the notation CQM-BEM-Elasticity will be adopted when an elasticity problem is solved employing the CQM-BEM procedure.

4. THE CQM APPLIED TO THE BEM: SCALAR APPROACH

The time-domain BEM integral equation that corresponds to the scalar wave equation in a 2D domain Ω with boundary $\Gamma = \Gamma_u \cup \Gamma_p$ can be written as (Dominguez, 1993):

$$c(\xi)u(\xi,t) = \int_{\Gamma_p}\int_0^{t^+} u^*(X,t;\xi,\tau)\, p(X,\tau) d\tau\, d\Gamma_p - \int_{\Gamma_u}\int_0^{t^+} p^*(X,t;\xi,\tau) u(X,\tau) d\tau\, d\Gamma_u \tag{19}$$

In the above expression the coefficient is only $c(\xi) = 1$ when the source point ξ belongs to Ω and $c(\xi) \neq 1$ otherwise, $u(X,t)$ represents the potential, $u^*(X,t;\xi,\tau)$ is the fundamental solution and its normal derivative is represented as $p^*(X,t;\xi,\tau) = \partial u^*(X,t;\xi,\tau)/\partial n$.

Note that in Eq. (19) the initial conditions related to the potential and its time derivative are considered null on $\Omega \cup \Gamma$. The boundary conditions can be given by:

$$\begin{cases} u(X,t) = \bar{u}(X,t), & X \in \Gamma_u \\ p(X,t) = \dfrac{\partial u(X,t)}{\partial n} = \bar{p}(X,t), & X \in \Gamma_p \end{cases} \tag{20}$$

where $\partial u(X,t)/\partial n$ represents the flux. The discretized version of Eq. (19) for each source point ξ when the CQM is applied is written as (Abreu, Carrer and Mansur, 2003):

$$c(\xi)\, u(\xi, t_n) = \sum_{j=1}^{E_{le}} \sum_{k=0}^{n} \boldsymbol{g}_{n-k}^{j}(r, \Delta t)\, p_k^j(X) - \sum_{j=1}^{E_{le}} \sum_{k=0}^{n} \boldsymbol{h}_{n-k}^{j}(r, \Delta t)\, u_k^j(X) \tag{21}$$

In Eq. (21) E_{le} is the number of elements Γ_j employed to approximate the boundary $(j = 1, 2 \ldots, E_{le})$, Δt is the time interval and $r = |X - \xi|$ represents the distance between ξ and X. Once a source point ξ is fixed for $t_n = n\Delta t$ time sampling points $(n = 0, 1, 2 \ldots, N)$, $u(\xi, t_n)$ can be computed by means of Eq. (21). The expressions of the quadrature weights $\boldsymbol{g}_n$ and $\boldsymbol{h}_n$, which permit the convolution with the discrete values $p_k^j(X)$ and $u_k^j(X)$ are:

$$\boldsymbol{g}_n^j(X, \xi, \Delta t) = \frac{\rho_r^{-n}}{L} \sum_{l=0}^{L-1} \int_{\Gamma_{pj}} \hat{u}^*(X, \xi, \frac{\gamma(\rho_r\, e^{il2\pi/L})}{\Delta t}) \Phi^j(X)\, d\Gamma_p\, e^{-inl2\pi/L} \tag{22}$$

$$\boldsymbol{h}_n^j(X, \xi, \Delta t) = \frac{\rho_r^{-n}}{L} \sum_{l=0}^{L-1} \int_{\Gamma_{pj}} \hat{p}^*(X, \xi, \frac{\gamma(\rho_r\, e^{il2\pi/L})}{\Delta t}) \Phi^j(X)\, d\Gamma_u\, e^{-inl2\pi/L} \tag{23}$$

Note that, in Eqs. (22) and (23), $\Phi^j(X)$ represents the interpolation functions employed in the boundary discretization, $\hat{u}^*(X, \xi, s)$ is the Laplace transform of the fundamental solution $u^*(X, \xi, t)$ and $\hat{p}^*(X, \xi, s)$ is the Laplace transform of $p^*(X, \xi, t)$. In Eq. (23), $p_k^j(X)$ and $u_k^j(X)$ are:

$$\begin{cases} p_k^j(X) = p^j(X, k\,\Delta t) \\ u_k^j(X) = u^j(X, k\,\Delta t) \end{cases} \quad X \in \Gamma_j \tag{24}$$

The transformed fundamental solution and its normal derivative are given by:

$$\begin{cases} \hat{u}^*(r,s) = 2\, K_0(sr) \\ \hat{p}^*(r,s) = \dfrac{\partial \hat{u}^*(r,s)}{\partial r} \dfrac{\partial r}{\partial n} = -2s\, K_1(sr)\, \dfrac{\partial r}{\partial n} \end{cases} \tag{25}$$

In Eq. (25) $K_0(sr)$ is the modified Bessel function of order zero and second type and $K_1(sr)$ is the modified Bessel function of first order and second type, see Abramowitz & Stegun, 1984. The next procedure to follow is the same that explained at section **2** for the computation of the final response on the time-domain $u(\xi, t_n)$.

In the present work, the notation CQM-BEM-Scalar will be adopted when a scalar problem is solved employing the CQM-BEM procedure.

5. IMPLEMENTATION OF THE CQM-BEM BASED ON FOURIER TRANSFORM

The Fourier transform of $\bar{y}(\omega)$ of a function of time $y(t)$ is defined as:

$$\bar{y}(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-i\omega t} dt \tag{26}$$

where ω is the Fourier transform parameter and $i = \sqrt{-1}$. The Discrete Fourier Transform (DFT) is given by:

$$\overline{y}(n/N)=\sum_{k=0}^{N-1} y(k\,\Delta t)\,e^{-i\,n\,l\,2\pi/L} \qquad (27)$$

where $k, n = 0,1,2\ldots, N-1$ and Δt is the sampling interval. An efficient computation of Eq. (27) is executed by the Fast Fourier Transform (FFT) algorithm (Cooley & Tukey, 1965).

The quadrature weights $\boldsymbol{g}_n$ depend on the integral of the Laplace transform of the fundamental solution and $\boldsymbol{h}_n$ depend on the integral of the Laplace transform of the normal derivative of the fundamental solution. Comparing Eqs. (12) and (13) given for the CQM-BEM-Elasticity problem and Eq. (22) and (23) of the CQM-BEM-Scalar problem with Eq. (27) it is seen that this quadrature weights can be accomplished in an efficient way by the FFT algorithm at points $n\Delta t$ (where $y(t)$ in Eq. (26) is zero for negative *t*).

In the CQM the supposition of $L = N$ (see Lubich 1988, 1988a) leads to an order of complexity of $O(N^2)$ for calculating the *N* coefficients of the quadrature weights. A direct calculation of Eq. (12)-(13) and Eq. (22)-(23) as an accumulated sum of products for each *n* would require consuming a great of cpu-time, especially when solving large-order time-dependent problems. This computational effort can be reduced to $O(N \log N)$ using the FFT technique.

6. NUMERICAL EXAMPLE

One application is presented next to compute numerically the displacements of a rod fixed at $x = a$, under a boundary condition of Heaviside-type, applied at $x = 0$, at $t = 0$ and kept constant from this time onwards, i.e., $\overline{p}(t) = p/E\,H(t-0)$ as shown in Fig. (1).

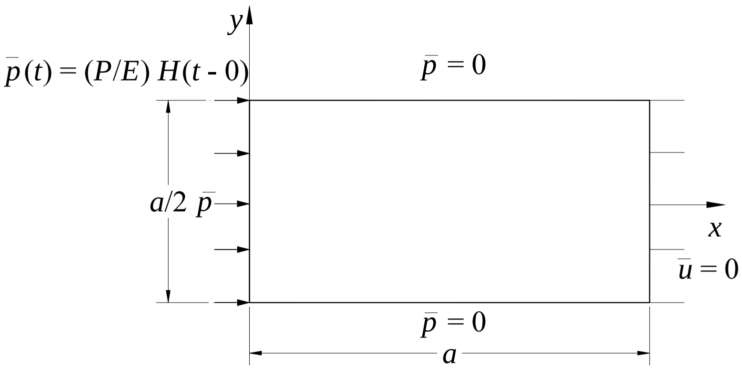

Figure 1: Rod under boundary condition prescribed.

In order to compute the numerical results, the approaches introduced in sections **3** and **4** are used: the first approach uses the CQM-BEM-Elasticity; the second one uses the CQM-BEM-Scalar. The implementations were compared with each other and with the analytical 1-D solution (Graff, 1975) with respect to their accuracy and computational efficiency.

In the CQM-BEM-Elasticity approach a Poisson´s coefficient $\nu = 0$ was used to avoid displacement variations on the *y*-direction allowing only longitudinal waves to exist. In this way a comparison with the CQM-BEM-Scalar approach is possible. Moreover, the velocity c_p is assumed to be equal 1.0 coinciding with the value of the scalar wave propagation velocity tested. The function $\gamma(z)$ adopted was the backward difference formula of second-order. The dimensionless parameter $\beta = c\,\Delta t / l$ was used to estimate the time-step length (*l* is the smallest boundary element length) employed in the analyses.

Figure (2) shows the boundary discretization used in both approaches as well as the boundary nodes *A*(0,0) and *B*(*a*,0) are selected for the comparison between numerical and analytical results: CQM-BEM-Elasticity approach interpolates the boundary unknowns with quadratic elements and linear elements were used to approximate the boundary variables in the CQM-BEM-Scalar case. The mesh tested possessed 76 nodes with 4 double nodes.

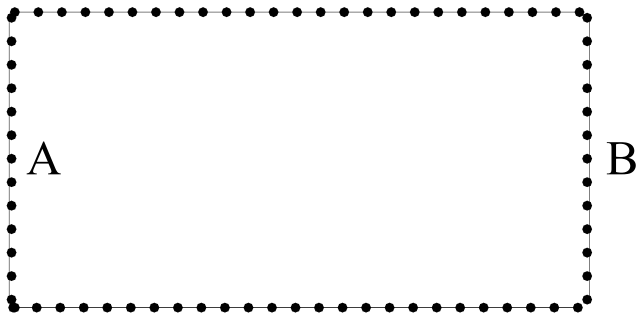

Figure 2: Rod under boundary condition prescribed: boundary discretization.

Figure (3) depicts two curves which represent the displacement time history at point *A* due to the boundary load $\overline{p}(t)$ and computed with the two approaches aforementioned. Tractions and flux time histories at the boundary node *B* are presented in Fig. (4).

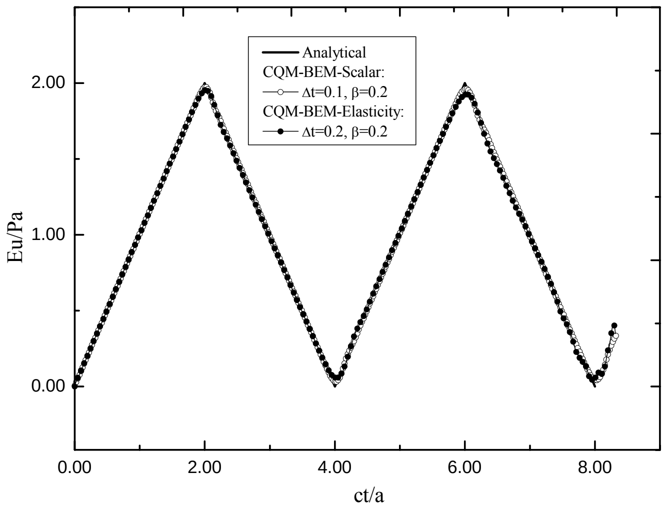

Figure 3: Displacements at point *A* for the rod under effects of the boundary load $\overline{p}(t)$.

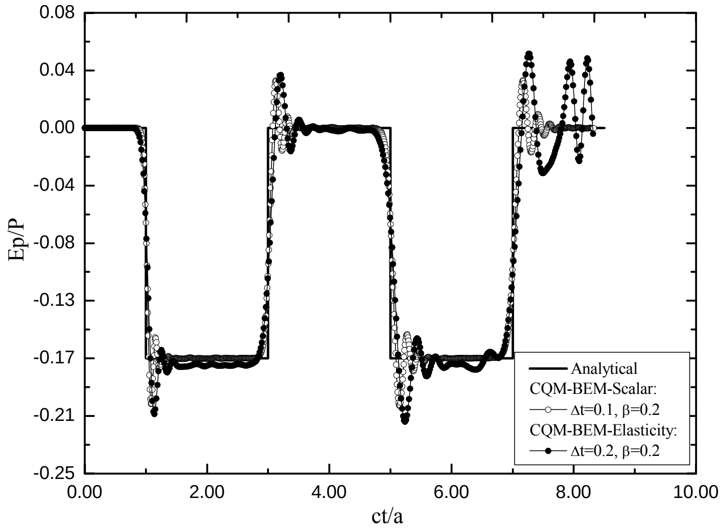

Figure 4: Tractions and flux time histories at the boundary node *B* for the rod under effects of boundary load $\overline{p}(t)$.

It is important to mention that Figs. 3 and 4 show the responses computed for an error $\delta = 10^{-4}$ when the CQM-BEM-Scalar approach was used and this error was assumed $\delta = 10^{-10}$ for the CQM-BEM-Elasticity approach. As expected (Lubich, 1988, 1988a), theses results demonstrate that small values of δ, e.g., $\delta = 10^{-10}$, lead to unstable results below which the results at the end of the analyses become a little different from those obtained with $\delta = 10^{-4}$. This kind of problem can be avoided for the CQM-BEM-Elasticity approach, reducing the value of δ. In what concerns numerical accuracy Δt can be as small as the machine accuracy allows, however, the authors experience demonstrated that excessively reduced Δt may not be accurate if the backward difference formula of second-order is employed. In this case, a backward difference formula of first-order should be employed.

In Table 1 required storage memory for the numerical implementations performed via CQM-BEM for both scalar and elasticity approaches is shown. These numerical calculations were carried out taking into account the dimension of the problem when the discretization in time was incremented for a refined fixed mesh (in this example, 76 nodes). The total time sampling values were selected as $N = 2^M$ $(M = 1, 2, 3, \ldots)$ and the tests were performed before and after the FFT algorithm implementation for these N different values of the Fourier coefficients. The available memory is the memory employed to store matrices $\boldsymbol{G}$ and $\boldsymbol{H}$ which represents almost the totality of the memory used in the numerical implementations. These matrices are complex matrices of the double complex type that store the quadrature weights $\boldsymbol{g}_n$ and $\boldsymbol{h}_n$ resultant from the integrations in the CQM-BEM-Scalar approach. In the CQM-BEM-Elasticity case this matrices correspond to ${}^n\boldsymbol{g}_{ij}$ and ${}^n\boldsymbol{h}_{ij}$.

A representative value for the number of Fourier coefficients $N = 1024$ was used to compute the available memory of the analyses via CQM-BEM; this value furnished good results determined by the relation of parameters β and Δt in both implementation.

Table 1: Availability of memory for analyzes via CQM-BEM

Approach	**Mesh**	**Required storage memory only for double complex matrices**
CQM-BEM-Scalar Matrix $\boldsymbol{G}$ depend on $\boldsymbol{g}_n$ Matrix $\boldsymbol{H}$ depend on $\boldsymbol{h}_n$	76 nodes 1 internal point	≈ 185.3MB
CQM-BEM-Elasticity Matrix $\boldsymbol{G}$ depend on ${}^n\boldsymbol{g}_{ij}$ Matrix $\boldsymbol{H}$ depend on ${}^n\boldsymbol{h}_{ij}$	76 nodes 1 internal point	≈ 370.6MB

A comparative study of both BEM procedures was performed in order to discuss the computational cost in each case. To verify this we plot cpu-time in seconds consumed by the implementations in Fig. (5) as a function of N. In this case it is possible to observe the advantage obtained with respect to the reduction of cpu-time in the implementations of the assemblage of matrices of quadrature weights when the FFT algorithm was implemented. This advantage is more distinctive in the CQM-BEM-Elasticity procedure. The implementation was in FORTRAN 90 and the cpu-time was obtained from an Intel Pentium IV.

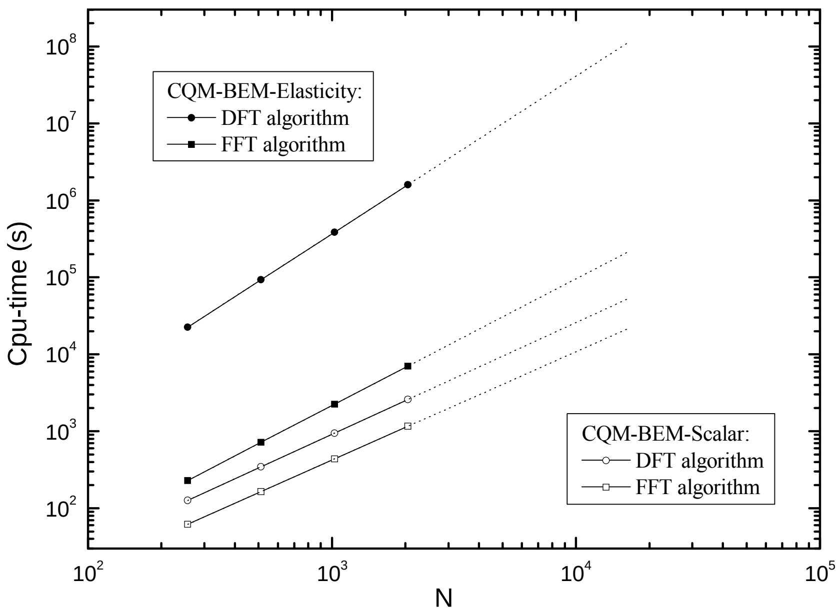

Figure 5: Cpu-time spent for computing response of the rod of Fig. (1) by two approaches via CQM-BEM versus the number of Fourier coefficients *N*.

7. CONCLUSION

In this paper a CQM-BEM formulation was implemented with a Fast Fourier Transform with the objective of reducing the computational cost. The FFT technique was introduced in the evaluation of the quadrature weights and it is important to note that the computational time was reduced by 50% in the example studied. In spite of this, other modifications could be implemented to further reduce the computational cost: note that until now the implementations made are only concerned with assemble of matrices ***H*** and ***G***, indispensable to perform the numerical convolution, but the actual numerical convolution can also be numerically improved. Also, the authors are working in the reduction of the $O(N)$ memory required to store matrices.

The bibliography shows that the CQM-BEM is a very robust and suitable technique to solve time-dependent problems when the Laplace transformed fundamental solution is known. The accuracy of the results and now the reduction of the computational cost are motivations to continue this multidisciplinary research. Efficiency tests must be complemented to adjust the technique developed.

8. ACKNOWLEDGEMENTS

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