A NOTE ON FLOW CLASSIFICATION

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Abstract. ***Astarita (1979) proposed a local and objective criterion to classify flows. His criterion is not restricted to MWCSH, ans is essentially an attempt to quantify the stress-relieving rotation experienced by the flowing material. Huilgol (1980) analyzed Astarita´s work and showed, through examples, some inconsistencies which rendered it useless as a general flow criterion. The present work revisits Huilgol´s examples and discusses in detaile the underlying physics that make Astarita´s criterion to fail for certain flows. This analysis leads to a new criterion for flow classification involving the concept of persistence of straining. A key kinematic entity introduced in the proposed criterion is the π-plane, a plane that is normal to the relative-rate-of-rotation vector. For a more comprehensive criterion, other parameters are needed in addition to a persistence-of-straining parameter. One of them is a measure of the deformation rate in the π-plane. Emphasis is given to isochoric motions. The proposed kinematic criterion is local, frame-indifferent and is not restricted to particular classes of flows. Its robustness is shown through detailed analyses of some examples of flows in the literature.***

Keywords. *Persistence-of-straining, flow-type, relative-rate-of-rotation, Non-Newtonian materials.*

1. Introduction

Non-Newtonian fluids are typically characterized by rheological functions which are determined through measurements during shear and (less often) extensional flows. However, the information in such measurements is rather incomplete if the mechanical behavior of a given material is to be thoroughly determined. Some materials exhibit elasticity and hence the stress at a material particle is a function of the history of deformation that it has experienced. Other materials are not isotropic, or are suspensions of rigid elongated particles. The stress in these materials is not expected to be a function of the deformation rate only. Therefore, a thorough characterization of a non-Newtonian material would require the measurement of rheological functions in different types of flow.

Many of the classical experiments that have been used to illustrate differences between Newtonian and Non-Newtonian behaviors, e.g., rod climbing, die swell, the tubeless siphon and the contraction flows--cannot be strictly classified under one single, well-defined type of flow. Actually, many of these are complex flows often referred to as ``predominantly viscometric,'' or ``predominantly extensional,'' or ``approximately rigid body motion.'' Moreover, typical engineering flows are also complex, i.e., the material often experiences a variety of types of flow as it moves along the process line.

Therefore, some important questions should be addressed. As just mentioned, there are some flows that are, say, “predominantly extensional”. However, for the sake of rigorousness, it would be quite useful to somehow quantify “how extensional” such flows are. For complex flows in general, it is also of practical importance from the rheological point of view to map as accurately as possible the regions of shear, extension, rigid body motion, etc.

The issue of flow classification is directly related to a key concept for the present paper, namely, the persistence-of-straining concept (Lumley (1969), Astarita (1979), Schunk and Scriven (1990), which is now briefly introduced. To this end, for simplicity let us employ the particular case of plane flows. We start by considering the eigenvalues and principal directions of the rate-of-strain tensor for these flows. In the absence of relative rotation, a material filament which is aligned, say, with the eigenvector corresponding to the largest positive eigenvalue, will be persistently stretched. On the other hand, it may happen that the fluid rotates in such a way that a different material filament is aligned with this eigenvector at each instant of time. In this case, a filament which is aligned with this eigenvector at some instant of time will subsequently rotate towards directions of less stretching. Thus, this material rotation relative to the eigendirections decreases the persistence of straining. It is worth noting that, when the two non-null eigenvalues are equal, then relative rotation will not affect the persistence of straining on the plane defined by the corresponding eigenvectors. Moreover, in general, the intensity of persistence of straining is a function of the difference between the non-null eigenvalues.

In this connection, two opposite extreme situations are of interest. One of them is the extensional flow (maximum persistence of straining). The other is the rigid body motion, i.e. a motion with no deformation, which is approached in a flow with finite rate of strain when the relative rate of rotation is high enough.

Astarita (1979) proposed that there are three properties that a representative criterion for flow classification should combine. It should be:

1. Local - it should indicate the flow type at each position in the flow.
2. Objective - it should be invariant under changes of reference frame.

3. Generally applicable - it should not be restricted to certain classes of flows.

Astarita (1979) also pointed out that a criterion can be either purely kinematic or can consider the response of the material to the flow (in addition to the kinematics). The three properties above are equally applicable to both types. In the present paper, we shall propose a purely kinematic criterion.

Huilgol (1980) analyzed Astarita's criterion and gave three examples which illustrate weaknesses of R_D as a flow classifier. The given examples show that: a) the parameter R_D does not hold a one-to-one correspondence with different flow types; b) when two eigenvalues of $\boldsymbol{D}$ have the same value, but are different from the third, there is no *a priori* reason to back up Astarita's choice for the relative-rate-of-rotation tensor and c) the limiting value for R_D when two of the eigenvalues approach each other can be different from the value directly calculated for two exactly equal eigenvalues.

The new criteria constructed does not present the short-comings of R_D. This will be clear by an application of the present criteria on examples similar to the ones given by Huilgol.

2. A new criterion for flow classification

An analysis of the classifier developed by Astarita (1979) and the criticism made by Huilgol (1980) have lead to conclusions of central importance in the construction of a new criterion for classification of flows. These inferences are now rephrased in a generalized form:

<u>Illation 1</u>: rotation cannot affect the strain which occurs orthogonally to the plane of relative rotation.
<u>Illation 2</u>: the change in the intensity of persistence of straining experienced by a material filament as it rotates relative to the eigenvectors of $\boldsymbol{D}$ is a decreasing function of the difference between the largest and smallest deformation rates that occur in the plane of relative rate of rotation.

Therefore, the first entity we need to define is a scalar quantity, I_n, which is the intensity of the rate of strain of an arbitrary material filament whose orientation at a given instant of time is denoted by the unit vector $\boldsymbol{e_n}$ It is given by:

$$I_n = \boldsymbol{e_n} \cdot \boldsymbol{D} \cdot \boldsymbol{e_n}$$

where $\boldsymbol{D}$ is the rate-of –strain tensor. The material derivative of I_n is:

$$\dot{I}_n = \dot{\boldsymbol{e}}_{\boldsymbol{n}} \cdot \boldsymbol{D} \cdot \boldsymbol{e_n} + \boldsymbol{e_n} \cdot \dot{\boldsymbol{D}} \cdot \boldsymbol{e_n} + \boldsymbol{e_n} \cdot \boldsymbol{D} \cdot \dot{\boldsymbol{e}}_{\boldsymbol{n}}$$

In this equation, $\dot{\boldsymbol{D}}$ is the material derivative of the rate-of-strain tensor. Let λ_1, λ_2 and λ_3 be the eigenvalues of $\boldsymbol{D}$ such that $\lambda_1 \geq \lambda_2$, $|\lambda_1| \geq |\lambda_3|$ and $|\lambda_2| \geq |\lambda_3|$; and let $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ be the corresponding unit eigenvectors. Thus, $\dot{\boldsymbol{D}}$ is given by:

$$\dot{\boldsymbol{D}} = \boldsymbol{D}^{\backslash} + \boldsymbol{\Omega} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{\Omega}\ , \quad \boldsymbol{D}^{\backslash} = \sum_{i=1}^{3} \dot{\lambda}_i \boldsymbol{e}_i \boldsymbol{e}_i$$

where $\boldsymbol{\Omega}$ is the tensor associated with the rotation of the eigenvectors of $\boldsymbol{D}$ through $\dot{\boldsymbol{e}}_i = \boldsymbol{\Omega} \cdot \boldsymbol{e}_i$. Because each material filament has a different rate of rotation, we adopt the vorticity as the angular velocity of $\boldsymbol{e_n}$, because it is a representative average rate of rotation of the material filaments composing the material element. Then we can write that $\dot{\boldsymbol{e}}_n = \boldsymbol{W} \cdot \boldsymbol{e_n}$ and the material derivative of I_n is rewritten as:

$$\dot{I}_n = \boldsymbol{e_n} \cdot (\boldsymbol{D}^{\backslash} + \boldsymbol{D}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{D}) \cdot \boldsymbol{e_n}$$

where $\overline{\boldsymbol{W}}$ is the relative-rate-of-rotation tensor defined through $\overline{\boldsymbol{W}} = \boldsymbol{W} - \boldsymbol{\Omega}$. We can break I_n into two parts, one of them, $\boldsymbol{e_n} \cdot \boldsymbol{D}^{\backslash} \cdot \boldsymbol{e_n}$, gives the rate of change of I_n due to changes with time of the eigenvalues of $\boldsymbol{D}$ following the material particle. The other part, $\dot{I}_n^*$, defined by:

$$\dot{I}_n^* = \boldsymbol{e_n} \cdot (\boldsymbol{D}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{D}) \cdot \boldsymbol{e_n}$$

gives the rate of change of I_n due to the relative rate of rotation. Therefore, $\dot{I}_n^*$ is the part of $\dot{I}_n$ that is related to the concept of persistence of straining.

It is now useful to define the relative-rate-of-rotation vector,

$$\overline{\boldsymbol{w}} = \boldsymbol{w} - \boldsymbol{\omega} = -\overline{\boldsymbol{W}} \cdot \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon}$ is the third-rank alternator tensor. When $\overline{w}$ is not null, this vector allows us to define the so-called π-plane, a plane which is normal to $\overline{w}$. The importance of this plane for the persistence-of-straining concept cannot be overemphasized, as it will become clear in the following discussion. The π-plane is a useful tool to circumvent the difficulties pointed out by Lumley (1969) regarding the many possible orientations of the vorticity relative to the eigenvectors of $\boldsymbol{D}$ of an arbitrary velocity field. A different possible path to circumvent these difficulties had already been discussed by Schunk and Scriven (1990), who proposed a decomposition of the relative rate of rotation.

Because of Illation 1, in the quest for a suitable measure of persistence of straining, it is appropriate to restrict our attention to directions $\boldsymbol{e_n}$ on the π-plane only. There are other three directions $\boldsymbol{e_x}$, $\boldsymbol{e_y}$ and $\boldsymbol{e_z}$, which are now defined, that play an important role on the persistence-of-strain parameter. Direction $\boldsymbol{e_z}$ is defined by $\overline{w}$, namely, $\boldsymbol{e_z} = \overline{\boldsymbol{w}}/\overline{w}$, where $\overline{w} = \sqrt{\overline{\boldsymbol{w}} \cdot \overline{\boldsymbol{w}}}$. Directions $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$ on the π-plane are defined as the directions of maximum and minimum values of I_n on that plane, respectively. The projection of tensor $\boldsymbol{D}$ on the π-plane is given by:

$$\boldsymbol{D_\pi} = \boldsymbol{D} - \left[\boldsymbol{e_z e_z} \cdot \boldsymbol{D} + \boldsymbol{D} \cdot \boldsymbol{e_z e_z} - \boldsymbol{e_z e_z} \cdot \boldsymbol{D} \cdot \boldsymbol{e_z e_z}\right] = \Lambda_x \boldsymbol{e_x e_x} + \Lambda_y \boldsymbol{e_y e_y}$$

It can be shown that $\boldsymbol{e_x}$ and $\boldsymbol{e_y}$ are the eigenvectors of $\boldsymbol{D_\pi}$ and Λ_x and Λ_y are the corresponding eigenvalues. Generally, $\dot{I}_n^*$ is a function of the angle between the considered filament and, say, direction $\boldsymbol{e_x}$.

2.1 The persistence-of-straining parameter

An average of $\dot{I}_n^*$, called here $\overline{\dot{I}_n^*}$, is calculated, between the directions of minimum and maximum values of I_n, in order to have a representative value of $\dot{I}_n^*$ at a material particle. It can be shown that, we can write, for that average the following expression:

$$\overline{\dot{I}_n^*} = \left(\Lambda_x - \Lambda_y\right)\overline{w}$$

It is worth noting that $\overline{\dot{I}_n^*}$ is always non-negative.

It is convenient to make the quantity $\overline{\dot{I}_n^*}$ dimensionless with the aid of appropriate local quantities of the flow. To this end, we choose to compare the difference between the eigenvalues of $\boldsymbol{D_\pi}$ with a measure of its intensity, namely, $\sqrt{tr\boldsymbol{D}_\pi^2}$, while the relative rate of rotation, $\overline{w}$, is compared with a measure of the intensity of the total rate of deformation, namely, $\sqrt{tr\boldsymbol{D}^2}$. The result is the parameter $\Re$, a new measure of persistence of straining:

$$\Re = \frac{\overline{\dot{\mathrm{I}}_\mathrm{n}^*}}{\sqrt{tr\boldsymbol{D}_\pi^2}\sqrt{tr\boldsymbol{D}^2}}$$

which is also always non-negative, and increases monotonically as the intensity of persistence of straining is decreased. It is worth noting that, in the construction of the persistence-of-straining parameter, $\Re$, we did not consider the contribution of $\boldsymbol{D}^\nabla$ to the rate of change of I_n. This is justified because for transient flows in which the flow type is fixed, it is clear that the eigenvalues of $\boldsymbol{D}$ do change in time. Therefore, as far as flow classification and the concept of persistence of straining are concerned, the only part of the material derivative of I_n that matters is $\boldsymbol{e_n} \cdot (\boldsymbol{D}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{D}) \cdot \boldsymbol{e_n}$. However, the complete material derivative of I_n can play a significant role in the understanding of general flows, and a detailed analysis of it is in course.

$\Re$ is a measure of the intensity of persistence of straining. The two opposite extreme situations mentioned earlier are the boundaries of this parameter. For extensional flows, which corresponds to maximum intensity of persistence of straining, $\Re = 0$. The other extreme, namely, the vicinity of rigid body motion, corresponds the limit of zero intensity of persistence of straining. In this case, $\Re \to \infty$.

This parameter is also a measure of the weakness of the flow --in the sense of Tanner and Huilgol (1975) classification-- or, equivalently, the inverse of $\Re$ is a measure of how strong the flow is. For non-null relative rates of rotation, we can combine the above equation with Eq.\ (\ref{eq:newst-re}), and $\Re$ becomes:

$$\Re = \frac{\Lambda_x - \Lambda_y}{\sqrt{\Lambda_x^2 + \Lambda_y^2}} \frac{\overline{w}}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$$

2.2. Other usefull parameters

In the previous section we proposed a persistence-of-straining parameter that considers the straining which occurs in the π-plane only, consistently with Illation 2. In order to construct a more complete criterion for flow classification, it is also important to somehow distinguish flows with different intensities of the strain rate that occurs off the π-plane. We now introduce two parameters that can give important information about the rate of deformation on the direction of the rate of deformation $\overline{\boldsymbol{w}}$ and on the π-plane.

The first one, G, compares the intensity of the strain rate occurring in the $\overline{\boldsymbol{w}}$ -direction with the intensity of the total strain rate. It is given by:

$$G = \frac{\boldsymbol{e}_z \cdot \boldsymbol{D} \cdot \boldsymbol{e}_z}{\sqrt{tr\boldsymbol{D}^2}}$$

This parameter is sensitive to the sign of the deformation rate in the $\overline{\boldsymbol{w}}$ -direction.

The second parameter, H, compares the intensity of the strain rate occurring on the π-plane with the intensity of the total strain rate is given by:

$$H = \frac{\sqrt{tr\boldsymbol{D}_{\boldsymbol{\pi}}^2}}{\sqrt{tr\boldsymbol{D}^2}}$$

It is clear that H can take only non-negative values. It is interesting to notice that, when H=1, then $\boldsymbol{D}=\boldsymbol{D}_{\boldsymbol{\pi}}$ and therefore the flow is (locally) a plane flow.

It is important to emphasize that G and H are not defined when $\overline{\boldsymbol{w}} = \boldsymbol{0}$.

3. Performance of the new criteria on some examples

3.1 Example 1

The flow considered in this example is:

$$\boldsymbol{v} = (a_1 x - wy)\boldsymbol{e_x} + (wx + a_2 y)\boldsymbol{e_y} + a_3 z$$

For this flow, the rate-of-strain tensor $\boldsymbol{D}$ and the vorticity tensor $\boldsymbol{W}$ are simply:

$$[\boldsymbol{D}] = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \quad \text{and} \quad [\boldsymbol{W}] = \begin{bmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the eigenvectors of $\boldsymbol{D}$ lie along the axes, they do not rotate and so $\overline{W} = W$. The expression for Astarita's classifier is:

$$R_D = \frac{2w^2}{a_1^2 + a_2^2 + a_3^2}$$

It is clear from the expression above that for this flow R_D can take any value in $[0, \infty)$. Huilgol (1980) points out that, in particular, we can choose $w^2 = 1/2\left(a_1^2 + a_2^2 + a_3^2\right)$, which makes R_D=1, but the corresponding flow is not viscometric. In the present criterion, de necessary and sufficient conditions for the flow to be viscometric are $\Re = 1$ and H=1. Since we are dealing with incompressible materials, another condition is tr $\boldsymbol{D}$ = 0.

$$H = \frac{a_1^2 + a_2^2}{a_1^2 + a_2^2 + a_3^2} = 1 \Rightarrow a_3 = 0$$

$$\text{tr}\boldsymbol{D} = a_1 + a_2 + a_3 = 0 \Rightarrow a_1 = -a_2$$

$$\Re = \frac{2|a_1|}{\sqrt{2a_1^2}} \frac{w}{\sqrt{2a_1^2}} = 1 \Rightarrow w = \pm a_1$$

It can be proven that these conditions yield to a viscometric flow.

3.2 Example 2

This example was adapted from Huilgol (1980) such as to illustrate the other two shortcomings of R_D mentioned earlier. Let us consider the isochoric flow given by the following velocity field:

$$\boldsymbol{v} = [(a + \varepsilon)x - wy]\boldsymbol{e}_x + [wx + (a - \varepsilon)y]\boldsymbol{e}_y - 2a\boldsymbol{e}_z$$

where ε is a real parameter. For ε=0, Huilgol's (1980) second example is recovered. For this flow, the rate-of-strain tensor $\boldsymbol{D}$ and the vorticity tensor $\boldsymbol{W}$ are simply:

$$[\boldsymbol{D}] = \begin{bmatrix} a + \varepsilon & 0 & 0 \\ 0 & a - \varepsilon & 0 \\ 0 & 0 & -2a \end{bmatrix} \quad \text{and} \quad [\boldsymbol{W}] = \begin{bmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the eigenvectors of $\boldsymbol{D}$ lie along the axes, they do not rotate and so $\overline{W} = W$. The expression for Astarita's classifier is:

$$R_D = \frac{w^2}{3a^2 + \varepsilon^2}$$

It is clear that R_D is not continuous in the vicinity of ε=0, as the flow approaches an extensional flow, as it is shown in the following result:

$$\lim_{\varepsilon \to 0} R_D(\varepsilon) = \frac{w^2}{3a^2} \text{ and } R_D(\varepsilon = 0) = 0. \text{ Therefore, } \lim_{\varepsilon \to 0} R_D(\varepsilon) \neq R_D(\varepsilon = 0)$$

4. Final Remarks

A new criterion for classification of flows is proposed. It is local, objective, and is not restricted to any class of flows. This criterion is based on a new persistence-of-straining parameter, $\Re$, but it is shown that for a more complete classification other parameters are needed.

The parameter $\Re$ and the persistence-of-straining parameter proposed by Astarita (1979), namely, R_D are equivalent for isochoric plane flows. Huilgol (1980) identified inconsistencies in Astarita's parameters when applied to other flows.

The criterion proposed here does not present such inconsistencies, as we elaborate next.

1. For the classical flow types, it yields the following one-to-one correspondences:

   Viscometric Flow $\Leftrightarrow (\Re = 1,\ H = 1)$
   Extensional Flows $\Leftrightarrow (\Re = 0)$
   Nearly rigid body motion $\Leftrightarrow (\Re \to \infty)$

2. It is not necessary to change the definition of the tensor $\boldsymbol{\Omega}$ when two eigenvalues of $\boldsymbol{D}$ are equal, since the tensor that arises in the criterion is not $\boldsymbol{\Omega}$ by itself, but $\boldsymbol{D} \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \boldsymbol{D}$, which is uniquely determined in this case.

3. $\Re$ does not present discontinuities, which are observed in R_D exactly because the former requires no special treatment when two eigenvalues of $\boldsymbol{D}$ are equal.

The criterion for flow classification presented in this paper has arisen from a detailed examination of the physics involved in the process of persistence of straining, and its successful application to different types of flows has demonstrated its robustness.

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