A 1 ½ FINITE ELEMENT LAYER MODEL FOR BAROCLINIC OCEAN CIRCULATION

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Abstract. *A finite element model based on a Petrov Galerkin formulation in space and time, is proposed to simulate the hydrodynamics of ocean circulation. The ocean model is a two dimensional gravity reduced layer model, that has an active layer overlaying a deep inert layer where the pressure gradient is set to zero. The Petrov-Galerkin formulation considered here, use stabilising operators to improve the classical Galerkin approaches. Numerical experiments in a schematised ocean are performed. The response of the coastal ocean forced by non-uniform wind fields shows typical wind-driven circulation features such as upwelling and hydrodynamic gyres.*

***Keywords: Ocean Circulation, Baroclinic motions, Finite elements, Petrov-Galerkin formulation, Stabilising operators*.**

1. Introduction

Most of the main features of the coastal ocean circulation problems are concerned with the hydrodynamics effects induced by wind fields. Coastline configuration, bottom topography and non uniform wind field forcing are important for the transient and meander-like dynamics in coastal seas associated with the presence of cool surface temperatures in form of plumes and intrusion of warm oceanic waters. There are numerous papers about coastal circulation in coastal ocean using analytical and numerical approaches. But features as the upwelling, downwelling, coastal jets and mixing are poorly resolved by numerical models due many problems associated to the parameterisations and strong variability of physical processes, the delimitation of the computational domain and the resolution of the models. The circulation near the coastal boundaries is highly variable in space and time. In the nature the coastline irregularities and the spatial variation of the wind stress field lead to a three-dimensional circulation. Winds over coastal regions are generally upcoast or downcoast exhibiting spatial variability with characteristic scales smaller than the synoptic scale of atmospheric pressure patterns (Enriquez and Friehe, 1995). Non uniform wind stress fields cause water convergence and divergence, leading to the Ekman pumping.

The numerical models are of value for conceptual studies under idealized considerations, but success in using numerical models for simulation of time-dependent w ind-driven real coastal flows has been rather limited. Some efforts toward the development of modelling capability for coastal flow fields were published since the 90's using finite difference methods (e.g. Allen et al. ,1995; Federiuk and Allen 1995; Allen and Newberger, 1996; Wang 1997; Weisberg et al. 2000; Middleton et al. 1998; Weisberg et al. 2000; Carbonel 1998, 2002).

Finite element methods in hydrodynamic problems were mostly applied to the shallow water problems since the 70's (e.g. Grotkop, 1973; Sundermann, 1977; Taylor and Davies, 1975). But the models based on classical Galerkin formulations have shown some misbehaviours in time-dependent propagation problems. It is only possible to obtain numerical schemes that retain the high accuracy of the element-based spatial discretization for small values of the time step Δt. For quite modest values of Δt, the accuracy and phase-propagation properties of the Galerkin formulation are lost and the stability range is reduced in comparison with finite difference schemes (Donea and Quartapelle, 1992).

These disadvantages have motivated in the last years, the development of alternatively finite element formulations such as Taylor-Galerkin, least-squares Galerkin and Petrov-Galerkin methods to improve the time-accuracy approximations of evolution problems, particularly for advection-dominated problems.

Finite element formulations based on the SUPG and similar operators proposed initially for advection-difusion and compressible flow problems (Brooks and Hughes, 1982; Hughes and Tezduyar, 1984; Johnson 1987; Hughes and Mallet, 1986; Hughes, 1987; Shakib, 1988; Galeão and Do Carmo, 1988; Almeida and Galeão, 1996) are well known and valid alternatively. Also, for shallow water application problems some progress were reported (Bova and Carey,1995; Saleri, 1995; Carbonel, Galeão and Loula, 1995; Ribeiro, Galeão and Landau, 1996).

In this paper a finite element model based on a space-time Petrov-Galerkin formulation is proposed to numerically describe the hydrodynamics of a baroclinic coastal ocean forced by non uniform wind fields in the southern hemisphere.

2. The Ocean Model

The vertical structure is very simple considering only one dynamic upper layer of density $\rho^u(x_1,x_2,t)$ with thickness h and an inert lower layer $\rho^l(x_1,x_2,t)$. The vertically integrated conservation equations eliminating the fast barotropic mode of the two layer system could be written in the following form:

$$\frac{\partial U_i}{\partial t}+\frac{\partial u_j U_i}{\partial x_j}+\varepsilon_{ij}U_j+gh\{\sigma\frac{\partial h}{\partial x_i}\}+rU_i-\frac{\tau_i}{\rho^u}=0 \tag{1}$$

$$\frac{\partial h}{\partial t}+\frac{\partial U_i}{\partial x_i}-w_e=0 \tag{2}$$

where

$$\varepsilon=\begin{pmatrix}0 & -f\\ f & 0\end{pmatrix} \quad \text{and} \quad w_e=\frac{(H_e-h)^2}{t_e H_e}$$

The entrainment $\mathbf{w}_e$ is different from zero when $h \leq H_e$, where H_e is the entrainment depth. In the present paper, we use the formulation proposed by McCreary and Kundu (1988) in the numerical experiments.

The flux components are represented by $U_i = u_i h$, and u_i is the velocity components in the upper layer. The entrainment velocity is w_e, and h is the upper layer thickness. The wind stress components are represented by τ_i. The parameter r is the Rayleigh friction coefficient representing the sum of all dissipative losses, and f is the Coriolis parameter. The parameters ρ^u, ρ^l, ρ_{air} are the densities in the upper layer, lower layer and air respectively. The parameters $\mu=\rho^u/\rho^l$, $\sigma = 1 - \mu$ are density ratios. The wind stress field represented by $\tau_1(t)$ and $\tau_2(t)$ are evaluated by $\tau_i = c_w \rho_{air} Wi\,|W|$, where W^i are the wind velocity components.

To solve the equations (1) and (2), the following boundary conditions are considered :

$u = v = 0$

at the closed boundaries. For the initial conditions, an appropriate initial state is necessary to be assumed in the domain Ω and at the boundary Γ:

$u = u^o,\ v=v^o,\ h= h^o \qquad at \qquad t = 0$

where u^o and h^o represent the initial velocity component and elevation respectively.

3. Finite Element Method

To present the finite element methods, it is advantageous to rewrite the governing equations (1) and (2) for the baroclinic two-dimensional problem in function of the variables u, v,. Here u, v are the velocity components d in a x_1,x_2 coordinate system respectively, and $d = 2c$ and $c=\sqrt{g\sigma h}$ is the baroclinic celerity . The equation system reads

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x_1}+v\frac{\partial u}{\partial x_2}+c\frac{\partial d}{\partial x_1}+ru-fv-\frac{\tau_x}{\rho^u h}=0 \tag{3}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x_1}+v\frac{\partial v}{\partial x_2}+c\frac{\partial d}{\partial x_2}+rv+fu-\frac{\tau_y}{\rho^u h}=0 \tag{4}$$

$$\frac{\partial d}{\partial t}+u\frac{\partial d}{\partial x_1}+v\frac{\partial d}{\partial x_2}+c\frac{\partial u}{\partial x_1}+c\frac{\partial v}{\partial x_2}-\frac{g\sigma}{c}w_e=0 \tag{5}$$

The equation system (4)-(5) written in matrix form reads

$$\frac{\partial V}{\partial t}+\underline{A}\frac{\partial V}{\partial x_1}+\underline{B}\frac{\partial V}{\partial x_2}+\underline{C}V+F=0 \tag{6}$$

where

$$\underline{A}=\begin{pmatrix} u & 0 & c \\ 0 & u & 0 \\ c & 0 & u \end{pmatrix}, \underline{B}=\begin{pmatrix} v & 0 & 0 \\ 0 & v & c \\ 0 & c & v \end{pmatrix}, \underline{C}=\begin{pmatrix} r & -f & 0 \\ f & r & 0 \\ 0 & 0 & 0 \end{pmatrix}, F=\begin{pmatrix} -\tau_x / \rho^u h \\ -\tau_y / \rho^u h \\ -w_e g\sigma / c \end{pmatrix}, V=\begin{pmatrix} u \\ v \\ d \end{pmatrix} \qquad (7)$$

3.1 Petrov-Galerkin formulations

We now construct now space-time Petrov-Galerkin finite element models for the above problem. To this end we introduce a space-time finite element partition $\pi^{h,\Delta t}$, in which the time interval is partitioned into subintervals

$$I_n = t_{n+1} - t_n = \Delta t \quad , \quad t_n \in (0,T] \qquad (8)$$

where t_n , t_{n+1} belong to an ordered partition of time levels $0 = t_0 < \ldots t_n < t_{n+1} \ldots < t_F = T$ an the space domain Ω is partitioned in N sub-domains Ω_e with boundary Γ_e. The space-time integration domain is the slab $S_n = \Omega \times I_n$ with boundary $\Gamma_n = \Gamma_e \times I_n$ such that the slab is composed by N elements $S_n^e = \Omega_e \times I_n$.

Let P_n^h the finite element space of piecewise polynomial in space and time on the slab S_n. Defining

$$U_n^h = \{V^h \in P_n^h \times P_n^h ; V^h = \overline{V}^h \ on\ \Gamma\} \qquad (9a)$$

$$\hat{U}_n^h = \{\hat{V}^h \in P_n^h \times P_n^h ; \hat{V}^h = 0 \ on\ \Gamma\} \qquad (9b)$$

We say that the general space-time Petrov-Galerkin approximate solution for the baroclinic problem (6) is the vector $V^h \in U_n^h$, which satisfies

$$\int_{S_n} \hat{V}^h L^h d\Omega\, dt + \sum_{e=1}^{N} \int_{S_n^e} \overline{\Psi}_e G^h L^h\, d\Omega\, dt$$
$$+ \int_{\Omega} \hat{V}^h(t_n^+)[V^h(t_n^+) - V^h(t_n^-)]\, d\Omega = 0 \qquad \forall\, \hat{V}^h \in \hat{U}_n^h \qquad (10)$$

where

$$L^h = \underline{I}\frac{\partial V^h}{\partial t} + \underline{A}(V^h)\frac{\partial V^h}{\partial x_1} + \underline{B}(V^h)\frac{\partial V^h}{\partial x_2} + \underline{C}(V^h) + F \qquad (11)$$

is the residual vector associated with (6) and

$$G^h = \underline{I}\frac{\partial \hat{V}^h}{\partial t} + \underline{A}(V^h)\frac{\partial \hat{V}^h}{\partial x_1} + \underline{B}(V^h)\frac{\partial \hat{V}^h}{\partial x_2} \qquad (12)$$

is a space-time operator. The first integral appearing in Eq. (10) is the space-time Galerkin residual. The second integral represents the added Petrov-Galerkin contribution, in which $\overline{\Psi}_e$ contains the stabilizing parameters (Hughes and Mallet, 1986). The third integral is a jumping term which enforces weakly the initial condition in S_n. A diagonal matrix $\overline{\Psi}_e = \gamma\, \underline{I}$ is adopted, where γ is the intrinsic time scale free parameter. Using these assumptions, Eq. (10) can be rewritten as

$$\sum_{e=1}^{N} \int_{S_n^e} \hat{p}^h\ L^h d\Omega dt + \int_{\Omega} \hat{V}^h(t_n^+)\,[V^h(t_n^+) - V^h(t_n^-)]\, d\Omega = 0 \qquad \forall \hat{V}^h \in \hat{U}_n^h \qquad (13)$$

where

$$\hat{p}^h = \hat{V}^h + \gamma \left[\frac{\partial \hat{V}^h}{\partial t} + \underline{A}(V^h)\frac{\partial \hat{V}^h}{\partial x_1} + \underline{B}(V^h)\frac{\partial \hat{V}^h}{\partial x_2} \right] \qquad (14)$$

is the Petrov-Galerkin weighting function.

3. 2 Space-time Petrov-Galerkin formulation (STPG).

The space time Petrov-Galerkin method (**STPG**) is applied to a baroclinic ocean circulation problem in this paper. Linear interpolation in time will be considered combined with linear interpolations in space. In this case, in the Eq. 10, for each time step, the initial condition is strongly enforced as the last computed solution at the end of the previous time-step. As a result the jumping term disappear and the STPG formulation reads

$$\sum_{e=1}^{N} \int_{S_n^e} \{\hat{V}^h + \gamma \left[\frac{\partial \hat{V}^h}{\partial t} + \underline{A}(V^h)\frac{\partial \hat{V}^h}{\partial x_1} + \underline{B}(V^h)\frac{\partial \hat{V}^h}{\partial x_2} \right]\}\{ \frac{\partial V^h}{\partial t} + \underline{A}(V^h)\frac{\partial V^h}{\partial x_1} + \underline{B}(V^h)\frac{\partial V^h}{\partial x_2} + \underline{C}(V^h) + F\} d\Omega \, dt = 0 \qquad (15)$$

4. Numerical experiments

The experiments evaluates the solution using a idealised representation of a non uniform wind field constant in time. Non uniform wind fields are described by the composition of patches of the form

$W_1 = W_{01}\psi_1(x_1,x_2)$, $W_2 = W_{02}\psi_2(x_1,x_2)$,

where $\psi_1(x_1,x_2), \psi_2(x_1,x_2)$ are the two-dimensional structures of the patches.

The basic parameter values used in the experiments are the following: The fluid in the layers is initially at rest. The Coriolis parameter is taken as $f = 1.21 \,.10^{-4}$ sec^{-1}. The wind flows to the south and the maximal velocity components are fixed equal to W_{01} = 0 m/s, W_{02} = - 15 m/s . The global friction coefficient is fixed at $r = 1.2\ 10^{-6}\ sec^{-1}$ and the time step is defined as Δt=21600sec. The value of $c_w = 2.5\ 10^{-3}$ is adopted . The densities of the upper and lower layer are ρ^u=1023 kg m^{-3} and ρ^l=1024 kg m^{-3}. The parameter γ is chosen equal to Δt/8 . The rectangular ocean is 800 km x 1600 km and is presented in Figure 1. The initial upper layer thickness is 50 m and the characteristic speed of the system is c = 0.7ms^{-1}. The computational domain is a rectangular coastal ocean of 800km wide (west-east direction) and 1600k length (north-south direction). The domain is composed by 6400 triangular linear elements (Figure 1). The solutions will be obtained for wind patch applied in the central part of the coastal west boundary. The model is integrated from a state of rest (no motion) for a period of 5 days.

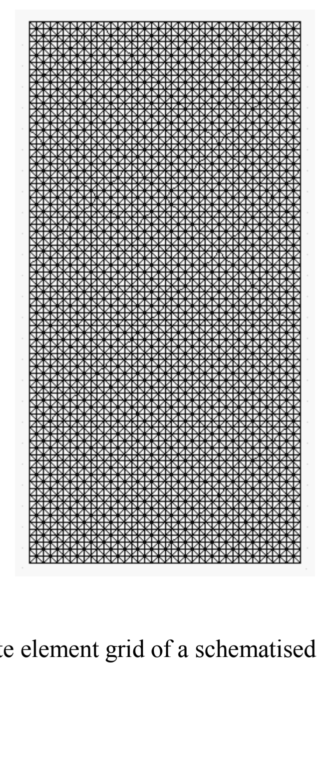

Figure 1. Spatial finite element grid of a schematised ocean of 800 km x 1600 km.

4.1 Coastal circulation driven by non uniform wind fields

4.1.1 Case 1. Southward wind

The coastal ocean is forced by a southward wind field. The wind field have a wide of 200km and extend southward along the coast 800km. The Figure 2 illustrates the response of the non-linear model to the non-uniform wind forcing, showing the velocity and the upper layer thickness fields at day 5. The response is characterised by a wind driven flow that decreases the upper layer thickness along a band in the west side coastline generating an upwelling band. The upper layer thickness decreases up to 7.8m. In the offshore side an area of larger thickness occur indicating a convergence zone. The velocity field show a strong flow divergence in the upwelling zone with velocities up to 45cms^{-1} near the coast diminishing in offshore direction. To determine the influence of the free parameter in the solution , we repeated the calculation using a larger free parameter (γ=Δt/4). The Figure 3 shows the upper layer thickness solution at day 5. There are a notable differences between the solutions. The thickness is significantly larger compared to the previous case. The upper layer thickness decreases only up to 15m. In this run the maximum velocity was 36 cms^{-1}.

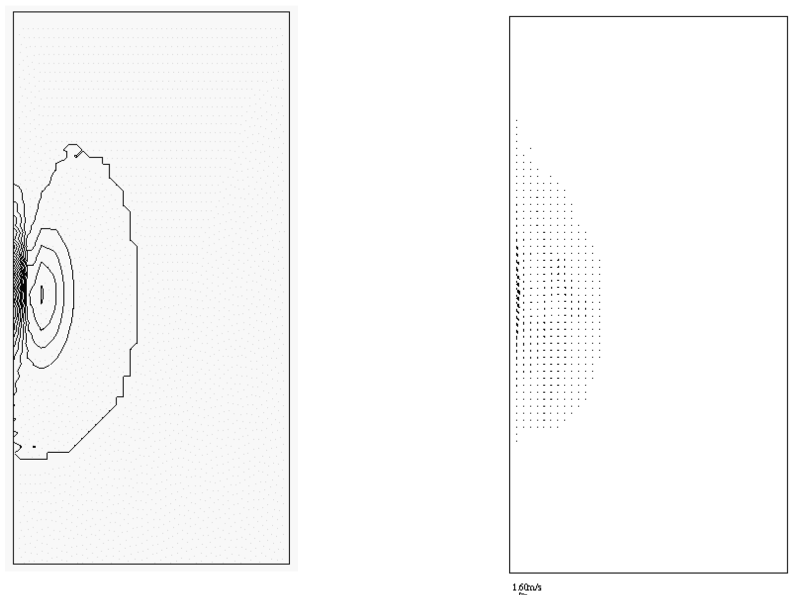

Figure 2. Upper layer thickness field (left panel) and velocity field (right panel) after 5 days in response to the southward wind patch. The free parameter is equal to Δt/8. The contour interval is 5m. The shallow thickness are located along the coast indicating upwelling, whereas the deeper thickness area is located offshore. The flow field shows a geostrophic component due Ekman pumping near the coast, decreasing offshore. The maximum velocity is 45cms^{-1}.

4.1.2 Case 2 Onshore winds

Strong onshore winds appears in the nature in different occasions due to atmospheric events, for example cold fronts. In this experiment, the coastal ocean is forced by a wind jet with a wide of 240km, extend eastward 400km. The wind patch is applied in the central part of the west coastline. The maximum wind velocity is 15ms^{-1}. Figure 4 shows the response at day 5 to the onshore wind forcing. Some properties of the resulting fields are the following. A current is directed onshore with a deflection to the south. This onshore drift forces a weak coastal upwelling in the northern side, thereby decreasing the thickness of the upper layer and generating a cell of shallower upper layer thickness. In the southern side, a convergence cell, where the upper layer thickness reaches the maximum thickness of 79 m due a strong convergence of the flows. The solution shows the initial generation of gyres.

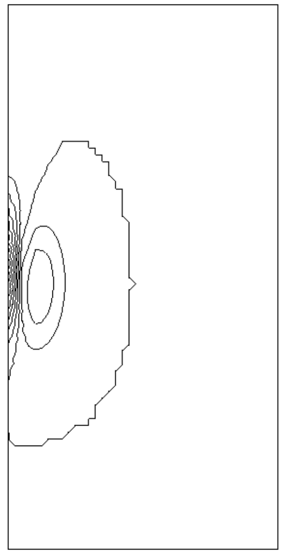

Figure 3. Upper layer thickness field after 5 days in response to the southward wind patch, when the free parameter is increased to Δt/4. The contour intervals are 5m. Similarly to the Figure 2, a divergence band is generated along the west boundary and a convergence cell if formed in the offshore side.

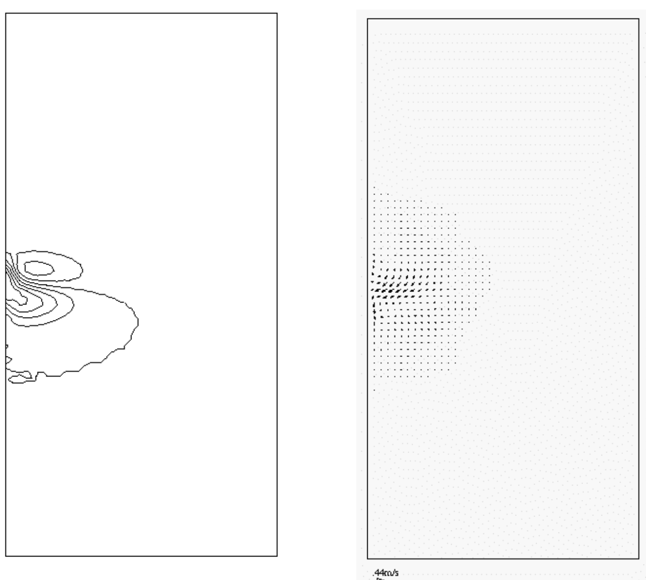

Figure 4. Upper layer thickness field (left panel) and velocity field (right panel) after 5 days in response to the onshore wind patch. The contour intervals are 5m. The solution shows two cells. One cell with centre at the coast and the another cell in the northern side. The maximal thickness of 79m is located in the coastal cell, whereas the minimum thickness of 37m is located in the northern cell. The maximum velocity is $12 cms^{-1}$

5. Summary and conclusions

A finite element model based on a Petrov Galerkin formulation in space and time, is proposed to describe the hydrodynamics of a coastal ocean with a simple vertical stratification. The ocean model is a two dimensional gravity reduced layer model, that has an active layer overlaying a deep inert layer where the pressure gradient is set to zero. The Petrov-Galerkin formulation considered here, use stabilising operators to improve the classical Galerkin approaches. A constant free parameter is considered in the experiment. Numerical experiments are performed considering a schematised ocean. The coastal ocean is forced by non-uniform wind fields. When a southward wind flows along the west boundary coast of the domain, the solution shows upwelling along the coast. A consequence of the divergence of the flows near the coastal boundary. When a westward wind flows in onshore direction, an onshore current results and gyres are generated.

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