MIXED FORMULATION FOR VISCOPLASTICITY

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Abstract. *The main objective of this paper is to propose a mixed Hellinger-Reissner variational formulation to deal with problems in which viscous phenomena take place beyond the elastic range, that is, the phenomena denoted in literature as elasto/viscoplasticity. The constitutive relation consider the von Mises yield criterion and Perzyna-like viscoplastic model. The paper shows that the solution of the equation system, defined by the equilibrium, kinematics and constitutive equation, are optimality conditions of an inf-sup mixed variational principle. Based on space discretization generated by the finite element method, by adopting a triangular finite element with quadratic and continuous interpolations for velocities and geometry and linear discontinuous interpolations for stress rate, a discrete version for the proposed mixed principle is also proposed. A numerical application is presented to validate the formulation.*

Keywords: *viscoplasticity, mixed formulation.*

1. Introduction

The main objective of this paper is to propose a mixed Hellinger-Reisssner variational formulation in which viscous phenomena take place beyond the elastic range, that is, the phenomena denoted in literature as elasto/viscoplasticity.

Viscoplasticity in metals is an important phenomenon when the absolute temperature exceeds one third of the absolute melting temperature. Nowadays there exists an increasing interest in that deformation process both under high and moderate temperatures because certain important materials also exhibit rate dependent deformation behavior at moderate temperatures. Additionally, failure processes in many engineering problems can be approached by adopting viscoplasticity models. As examples of these processes, one can mention for metals, propagation of Lüders bands and PortevinLe Chatelier effect, and for geo-materials shear bands and creep. (Heeres et al., 2002).

This paper regards viscoplasticity in its own right, but it is worth remembering that it can also be used as a regularized model for rate-independent plasticity. For example, the pure plasticity models usually fail both in perfect plasticity materials and in strain softening situation, because of the strain localization phenomenon. (Díez et al., 1998; Sluys, 1998).

Mixed formulations are an alternative to the reduced integration techniques for facing the locking phenomenon that happens in models of plastic materials complying with the Von Mises or Tresca yield criteria, when the Finite Element is applied. If the interpolation functions are not suitably chosen, the locking characteristics might lead to failure of the finite element method (Belytschko et al., 2000; Hughes, 1987). This paper will not discuss this aspect in details, but one should observe that this is the main motivation for choosing the mixed approach.

A triangular finite element with quadratic and continuous interpolations for velocities and geometry and linear discontinuous interpolations for stress rate is proposed. The element, herein proposed in the elasto/viscoplasticity context, comes from large experience with it in limit analysis applications and thermo-elasticity in incompressible materials (Borges et al., 1995; Costa and Borges, 2002).

The outline of this paper is as follows. Section **2** presents the concepts and assumptions used in the definition of the general principles that govern the behavior of isotropic bodies, constituted of elasto/viscoplastic material and subject to quasi-static load rates. The variational principles to describe the infinitesimal elasto/viscoplasticity problems are proposed in Section **3**. It is shown that the solution of the equation system, defined by the equilibrium, kinematics and constitutive equation, are optimality conditions of an *inf-sup* mixed variational principle. Finally, in Section **4**, based on space discretization generated by the finite element method and on the variational formulation of Section **3**, the discrete mixed principle is presented. In Section **5** a numerical application is presented to validate the formulation.

2. General Principles

This section contains a brief description of the continuum formulations for kinematics, equilibrium and constitutive relations for elasto/viscoplastic bodies subjected to quasi-static load rate.

2.1. Kinematic and Equilibrium

Consider a body occupying an open bounded region $\mathcal{B}$ with a regular boundary Γ at instant t. Let V denote the function space of all admissible velocity fields $\mathbf{v}$, sufficiently regular, complying with boundary conditions prescribed on a part Γ_v of Γ.

The strain rate tensor fields $\dot{\mathbf{E}}$ are elements of the function space W and the tangent deformation operator $\mathcal{D}$ maps V onto W

$$\dot{\mathbf{E}} = \mathcal{D}\,\mathbf{v} \qquad \forall\,\mathbf{v} \in V \tag{1}$$

Let $W^{'}$ be the space of stress rate fields $\dot{\mathbf{T}}$. The internal power for any pair $\dot{\mathbf{T}} \in W^{'}$ and $\dot{\mathbf{E}} \in W$ is defined by the duality product

$$\left\langle\, \dot{\mathbf{T}}\,,\, \dot{\mathbf{E}}\, \right\rangle = \int_{\mathcal{B}} \dot{\mathbf{T}} \cdot \dot{\mathbf{E}}\; d\mathcal{B} \tag{2}$$

Likewise, $V^{'}$ is the space of load rates and the external power which is dissipated by a load rate system $\dot{\mathbf{F}} \in V^{'}$ on a velocity field $\mathbf{v} \in V$ is given by the duality product

$$\left\langle\, \dot{\mathbf{F}}\,,\, \mathbf{v}\, \right\rangle = \int_{\mathcal{B}} \dot{\mathbf{b}} \cdot \mathbf{v}\; d\mathcal{B} + \int_{\Gamma_\tau} \dot{\mathbf{a}} \cdot \mathbf{v}\; d\Gamma \tag{3}$$

where $\dot{\mathbf{b}}$ and $\dot{\mathbf{a}}$ are body and surface load rates, respectively. Surface Γ_τ is the region of Γ where traction rates are prescribed. The boundary Γ will always consists of two disjoint parts, such that, $\Gamma = \Gamma_v \cup \Gamma_\tau$ and $\Gamma_v \cap \Gamma_\tau$ is empty.

The equilibrium condition, relating a stress rate field and a load rate system $\dot{\mathbf{F}} \in V^{'}$, is imposed by the Principle of Virtual Power

$$\left\langle\, \dot{\mathbf{T}}\,,\, \mathcal{D}\,(\mathbf{v}^* - \mathbf{v})\, \right\rangle \;=\; \left\langle\, \dot{\mathbf{F}}\,,\, (\mathbf{v}^* - \mathbf{v}) \right\rangle \quad \forall\,\mathbf{v}^* \;\in V \tag{4}$$

2.2. Constitutive relations

The hardening viscoplasticity material behaviour is presented in the framework of thermodynamic with internal variable. This approach is generally adopted because it provides the most natural way to develop variational principles in mechanics. This theory postulates the existence of an accompanying equilibrium state in the irreversible problem of evolution. Therefore, one assumes that evolution equations are expressed in terms of a potencial, or at least, of a pseudo-potencial , described by means of the internal variable rates.

For viscoplastic models, in quasi-estatic and isothermal processes, the only independent variable is the total strain $\mathbf{E}$. The internal variables, which describe the current state of the material, are the viscoplastic strain $\mathbf{E}_{vp}$ and another one, denoted $\boldsymbol{\chi}$, modeling the hardening deformation. The components of the hardening variable are the scalar isotropic variable χ_{iso}, associated with the dislocation density, and the tensorial kinematic variable $\boldsymbol{\chi}_{kin}$ associated with the incompatibility of various viscoplastic strains (Alfano et al., 2001; Angelis, 2000).

For small strains , the hypothesis of the partition of total strain into an elastic (reversible) and a viscoplastic (irreversible) strain is adopted, that is

$$\mathbf{E} \;=\; \mathbf{E}_e \;+\; \mathbf{E}_{vp} \tag{5}$$

Under the hypothesis of the partition of total strain, Eq. (5), the thermodynamic potential depends on the elastic strain $\mathbf{E}^e$ and the internal variables $\boldsymbol{\chi}$. In this context, it is permitted to consider the uncoupled between the specific elastic free energy and the hardening free energy (Lemaitre and Chaboche, 1994; Ulm and Coussy, 2003), or equivalently

$$\Psi(\mathbf{E}_e, \boldsymbol{\chi}) = \Psi_e(\mathbf{E}_e) + \Psi_{vp}(\boldsymbol{\chi}) \tag{6}$$

In case linear elasticity is adopted, the specific elastic energy $\Psi_e(\mathbf{E}^e)$ is ruled by the differential potential

$$\Psi_e(\mathbf{E}_e) = \frac{1}{2}\,\mathbb{D}\,\mathbf{E}_e \cdot \mathbf{E}_e \tag{7}$$

with

$$\mathbb{D} = \frac{E}{(1+\nu)}\,\mathbb{I} + \frac{E\,\nu}{(1+\nu)(1-2\nu)}\,(\mathbf{I}\otimes\mathbf{I}) \tag{8}$$

as $\mathbb{I}$ and $\mathbf{I}$ being the fourth and second identity tensors, respectively. The elastic constants are the Young's modulus E and the Poisson's ratio ν.

Under the additive strain decomposition (5), the functional in Eq. (7) can be recast in the following form

$$\Psi_e(\mathbf{E}, \mathbf{E}_{vp}) = \frac{1}{2}\mathbb{D}\,(\mathbf{E}-\mathbf{E}_{vp})\cdot(\mathbf{E}-\mathbf{E}_{vp}) \tag{9}$$

The hardening potential $\Psi_{vp}(\boldsymbol{\chi})$ is expressed by

$$\Psi_{vp}(\boldsymbol{\chi}) = \frac{1}{2}\,\mathbf{H}\,\boldsymbol{\chi}\cdot\boldsymbol{\chi} \tag{10}$$

where $\mathbf{H}$ is the hardening matrix defined as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{kin} & 0 \\ 0 & H_{iso} \end{bmatrix} \tag{11}$$

where H_{iso} and $\mathbf{H}_{cin}$ are defined as the isotropic and kinematic hardening module, respectively.

The stress is the generalized forces thermodynamical associated to the elastic strain $\mathbf{E}_e$ that is

$$\mathbf{T} = \nabla_{\mathbf{E}_e}\Psi = \nabla_{\mathbf{E}}\Psi = \mathbb{D}\mathbf{E}_e \tag{12}$$

The associated thermodynamical forces, which define the estate equations for internal variables, are

$$\mathbf{T} = -\nabla_{\mathbf{E}_{vp}}\Psi = \mathbb{D}\mathbf{E}_e \tag{13}$$

$$\mathcal{A}_{iso} = -\nabla_{\chi_{iso}}\Psi = -H_{iso}\chi_{iso} \qquad \text{and} \qquad \boldsymbol{\mathcal{A}}_{kin} = -\nabla_{\chi_{kin}}\Psi = -\mathbf{H}_{kin}\boldsymbol{\chi}_{kin} \tag{14}$$

The first variable of Eq. (14), $\mathcal{A}_{iso}$, represents the size of the existing elasticity domain and the second one, $\boldsymbol{\mathcal{A}}_{kin}$, the distance center of this domain from the origin. Both variables are components of the generalized thermodynamic force vector, noted here as $\boldsymbol{\mathcal{A}}$.

By applying the Legendre-Fenchel transformation to the potential Ψ, it is possible to define a dual potential Ψ^c(Lemaitre and Chaboche, 1994; Panagiotopoulos, 1985). This potential is dependent on the stress $\mathbf{T}$, the viscoplastic strain $\mathbf{E}_{vp}$ and the thermodynamic force $\boldsymbol{\mathcal{A}}$. Because of the conditions adopted in the free energy definition, the uncoupled between elastic and hardening part of the specific dual free energy is also accepted, that is

$$\Psi^c(\mathbf{T}, \mathbf{E}_{vp}, \boldsymbol{\mathcal{A}}) = \Psi^c_e(\mathbf{T}, \mathbf{E}_{vp}) + \Psi^c_{vp}(\boldsymbol{\mathcal{A}}) \tag{15}$$

For linear elasticity, the dual elastic energy $\Psi^c_e(\mathbf{T}, \mathbf{E}_{vp})$ reduced to the positive definite quadratic form (Costa and Borges, 2002)

$$\Psi^c_e(\mathbf{T}, \mathbf{E}_{vp}) = \frac{1}{2}\,\mathbf{T}\cdot\mathbb{D}^{-1}\mathbf{T} + \mathbf{T}\cdot\mathbf{E}_{vp} \tag{16}$$

with

$$\mathbb{D}^{-1} = \frac{(1+\nu)}{E}\,\mathbb{I} - \frac{\nu}{E}\,(\mathbf{I}\otimes\mathbf{I}), \tag{17}$$

The dual hardening potential is expressed by

$$\Psi^c_{vp}(\boldsymbol{\mathcal{A}}) = \frac{1}{2}\,\mathbf{H}^{-1}\boldsymbol{\mathcal{A}}\;\cdot\boldsymbol{\mathcal{A}} \tag{18}$$

here $\mathbf{H}^{-1}$ is the inverse of the hardening matrix $\mathbf{H}$.

The constitutive relations, inverse forms of $(12)_1$ and (14), are

$$\mathbf{E} = \nabla_{\mathbf{T}}\Psi^c = \mathbf{E}_{vp} + \mathbb{D}^{-1}\mathbf{T} \tag{19}$$

$$\chi_{iso} = -\nabla_{\mathcal{A}_{iso}}\Psi^c = -H_{iso}^{-1}\mathcal{A}_{iso} \qquad \text{and} \qquad \boldsymbol{\chi}_{kin} = -\nabla_{\mathcal{A}_{kin}}\Psi^c = -\mathbf{H}_{kin}^{-1}\boldsymbol{\mathcal{A}}_{kin} \tag{20}$$

The potential for rates is defined by

$$\mathcal{J}(\dot{\mathbf{E}}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) = \frac{1}{2}\mathbb{D}(\dot{\mathbf{E}} - \dot{\mathbf{E}}_{vp}) \cdot (\dot{\mathbf{E}} - \dot{\mathbf{E}}_{vp}) + \frac{1}{2}\mathbf{H}\,\dot{\boldsymbol{\chi}} \cdot \dot{\boldsymbol{\chi}} \tag{21}$$

such that the constitutive relations for the rates are

$$\dot{\mathbf{T}} = \nabla_{\dot{\mathbf{E}}}\mathcal{J} = -\nabla_{\dot{\mathbf{E}}_{vp}}\mathcal{J} = \mathbb{D}\dot{\mathbf{E}}_e - \qquad \dot{\boldsymbol{\mathcal{A}}} = -\nabla_{\dot{\boldsymbol{\chi}}}\mathcal{J} = -\mathbf{H}\dot{\boldsymbol{\chi}} \tag{22}$$

The dual of the potential for rates is also obtained by the Legendre-Fenchel transformation, leading to

$$\mathcal{J}^c(\dot{\mathbf{T}}, \dot{\Theta}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\mathcal{A}}}) = \frac{1}{2}\,\mathbb{D}^{-1}\dot{\mathbf{T}} \cdot \dot{\mathbf{T}} + \dot{\mathbf{T}} \cdot \dot{\mathbf{E}}_{vp} + \frac{1}{2}\mathbf{H}^{-1}\dot{\boldsymbol{\mathcal{A}}} \cdot \dot{\boldsymbol{\mathcal{A}}} \tag{23}$$

and

$$\dot{\mathbf{E}} = \nabla_{\dot{\mathbf{T}}}\mathcal{J}^c = \dot{\mathbf{E}}_{vp} + \mathbb{D}^{-1}\dot{\mathbf{T}} + \qquad \dot{\boldsymbol{\chi}} = -\nabla_{\dot{\mathcal{A}}}\mathcal{J}^c = -\mathbf{H}^{-1}\dot{\boldsymbol{\mathcal{A}}} \tag{24}$$

The same rate constitutive relations (22) and (24) can be obtained by deriving (12) , (14),(19) and (20) with respect to time.

2.2.1. Evolution Equation for Viscoplasticity

Viscoplasticity may be viewed as resulting from the optimality condition of the unconstrained function which appears as a regularized version of the Principle of Maximum Plastic Dissipation. (Angelis, 2000). In plasticity, the Principle of Maximum Plastic Dissipation is given by

$$D_p(\dot{\mathbf{E}}_p, \dot{\boldsymbol{\chi}}) = \sup_{(\mathbf{T}, \mathcal{A}) \in P} \{\mathbf{T} \cdot \dot{\mathbf{E}}_p + \boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\chi}}\} \tag{25}$$

where $\dot{\mathbf{E}}^p$ is the plastic strain rate and the set P defines the space of plastically admissible stress by

$$P \equiv \{(\mathbf{T}, \boldsymbol{\mathcal{A}}) \quad | \quad f(\mathbf{T}, \boldsymbol{\mathcal{A}}) \leq 0\} \tag{26}$$

where $f(\mathbf{T}, \boldsymbol{\mathcal{A}})$ is the yield function.

In viscoplasticity when the deformation process is elastic then $f(\mathbf{T}, \boldsymbol{\mathcal{A}}) \leq 0$, but if there is plastic loading , to the contrary of rate independent plasticity, $f(\mathbf{T}, \boldsymbol{\mathcal{A}})$ may be positive. Therefore, to describe the viscoplastic constitutive equation, the constrained maximum problem, Eq. (25), is transformed into an unconstrained one, by adding to the objective function a penalty function $\Phi_n^+ : I\!R \to I\!R^+$ of the constrained $f(\mathbf{T}, \mathbf{A}) \leq 0$ amplified by a penalty parameter η. In this way, the dissipation function is expressed as

$$D_{vp}(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) = \sup_{(\mathbf{T}, \mathcal{A})} \left\{\mathbf{T} \cdot \dot{\mathbf{E}}_{vp} + \boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\chi}} - \eta\, \Phi_n^+ \left(f(\mathbf{T}, \boldsymbol{\mathcal{A}}\right)\right\} \tag{27}$$

where the parameter $\eta \in (0, +\infty)$ represents a viscosity coefficient. The notation $(\,)^+$ refers to $(y)^+ \; = y\,H(y)$, where H is the Heaviside step function and n is a material constant(Alfano et al., 2001). It is worth observing that Φ_n^+, in such way defined, satisfies the condition $\Phi_n^+(x) = 0$ if and only if $x \leq 0$ and is non-negative in $I\!R$. Additionally, the penalty function Φ_n^+ of the constraint $f(\mathbf{T}, \boldsymbol{\mathcal{A}})$ needs to be of class $\mathcal{C}^1$(Angelis, 2000; Alfano et al., 2001).

When these conditions are fulfilled the dissipation potential is convex, is minimum at $(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) = 0$ and the optimality conditions for the unconstrained problem will be

$$\mathbf{T} \in \partial_{\dot{\mathbf{E}}_{vp}} D_{vp}(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \qquad \boldsymbol{\mathcal{A}} \in \partial_{\dot{\chi}} D_{vp}(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \tag{28}$$

and there will be a dual potential $D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}})$ such that

$$\dot{\mathbf{E}}_{vp} \in \partial_{\mathbf{T}} D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}}) \qquad \dot{\boldsymbol{\chi}} \in \partial_{\mathcal{A}} D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}}) \tag{29}$$

where

$$D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}}) := \sup_{\mathbf{E}^*_{vp}, \dot{\chi}^*} [\mathbf{T} \cdot \mathbf{E}_{vp}{}^* + \boldsymbol{\mathcal{A}} \cdot \dot{\boldsymbol{\chi}}^* - D_{vp}(\dot{\mathbf{E}}^{vp*}, \dot{\boldsymbol{\chi}}^*)] \tag{30}$$

Equations (28) and (29) define the internal variables evolutive law for the viscoplastic constitutive model and they have been known in literature as the Normality Law.

Since then the penalty function Φ^+_n is of class $\mathcal{C}^1$, it is sufficient the yield function $f(\mathbf{T}, \mathbf{A})$ to be differentiable to assure the dissipation D_{vp} differentiability. In this case, the dual dissipation D^c_{vp} also is differentiable and equal to the penalty function, that is

$$D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}}) = \eta\, \Phi^+_n[f(\mathbf{T}, \boldsymbol{\mathcal{A}})] \tag{31}$$

and the flux law (29) can be recast in the form

$$\dot{\mathbf{E}}_{vp} = \eta\, \frac{d\Phi^+_n}{df}(f)\, \nabla_{\mathbf{T}} f(\mathbf{T}, \boldsymbol{\mathcal{A}}) \qquad \dot{\boldsymbol{\chi}} = \eta\, \frac{d\Phi^+_n}{df}(f)\, \nabla_{\mathcal{A}} f(\mathbf{T}, \boldsymbol{\mathcal{A}}) \tag{32}$$

3. Mixed variational principle for infinitesimal elasto/viscoplasticity

The objective of this chapter is to propose a variational principle to describe elasto/viscoplasticity problems. One can see that the field solution of the equation system, defined by equilibrium, kinematics and constitutive equations, are optimality conditions of an $inf - sup$ mixed variational principle.

The elasto/viscoplasticity problem consists in determining paths of displacement $\mathbf{u}(t)$, stress $\mathbf{T}(t)$ and strain $\mathbf{E}(t)$, developed in an elasto/viscoplasticity body during a load program. If, at a moment t of the process, one considers all state variable $(\mathbf{E}, \mathbf{E}_{vp}, \boldsymbol{\chi})$ as known, then, from the constitutive relations (12) and (14), the dual variables $(\mathbf{T}, \boldsymbol{\mathcal{A}})$ might be able to be determined. Therefore, the next step will be obtaining the stress and strain rate fields that occur in the body when it is submitted to variation in force system $\mathbf{F}$ or/and in the displacement constraints $\mathbf{u}$, during a time interval dt.

In turn, at each moment, this problem consists in finding a stress rate field $\dot{\mathbf{T}} \in W'$, a kinematic hardening rate field $\dot{\boldsymbol{\chi}}_{kin} \in I\!R^n \,\mathrm{x}\, I\!R^n$, a isotropic hardening rate field $\dot{\boldsymbol{\chi}}_{iso} \in I\!R$, a strain rate field $\dot{\mathbf{E}} \in W$, a viscoplastic strain rate field $\dot{\mathbf{E}}^{vp} \in W$ and a velocity field $\mathbf{v} \in V$, such as the following equation system holds

$$(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \in \partial_{(\mathbf{T},\mathcal{A})} D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}}) \tag{33}$$

$$\dot{\mathbf{E}} = \mathcal{D}\,\mathbf{v} \qquad \forall\, \mathbf{v} \in V \tag{34}$$

$$\langle\, \dot{\mathbf{T}}\,,\, \mathcal{D}\,(\mathbf{v}^* - \mathbf{v})\,\rangle \;=\; \langle\, \dot{\mathbf{F}}\,,\, (\mathbf{v}^* - \mathbf{v})\rangle \quad \forall\, \mathbf{v}^* \in V \tag{35}$$

$$(\dot{\mathbf{T}}, -\dot{\boldsymbol{\mathcal{A}}}) = \nabla_{(\dot{\mathbf{E}},\dot{\chi})} \mathcal{J}(\dot{\mathbf{E}}, \dot{\mathbf{E}}^{vp}, \dot{\boldsymbol{\chi}}) \qquad \Longleftrightarrow \qquad (\dot{\mathbf{E}}, -\dot{\boldsymbol{\chi}}) = \nabla_{(\dot{\mathbf{T}},\dot{\mathcal{A}})} \mathcal{J}^c(\dot{\mathbf{T}}, \dot{\mathbf{E}}^{vp}, \dot{\boldsymbol{\mathcal{A}}}) \tag{36}$$

For a tridimensional continuum, under infinitesimal strain assumption, the tangent deformation operator $\mathcal{D}$, matches the symmetric part of the gradient ∇^s. The potentials $\mathcal{J}$ and $\mathcal{J}^c$ are defined by Eqs. (21) and (23), respectively. The dual dissipated function that defines the internal variable evolutive laws, Eq. (33), can be expressed by Eq. (31).

In the following part, one can see that a solution for this system is also the solution for a mixed variational principle, defined in function of velocity, stress rate and hardening rate fields.

The fields $(\dot{\mathbf{T}}, \dot{\boldsymbol{\mathcal{A}}})$, solutions for this system, are associated with the total and viscoplastic strain rate fields and with the kinematic and isotropic hardening rate fields by the constitutive relation (36). Then by the gradient definition

$$\langle\, \mathcal{J}(\dot{\mathbf{E}}^*, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}})\rangle - \langle \dot{\mathbf{T}}\,,\, \dot{\mathbf{E}}^*\rangle \;\geq\; \langle\, \mathcal{J}(\dot{\mathbf{E}}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}})\rangle - \langle \dot{\mathbf{T}}\,,\, \dot{\mathbf{E}}\rangle \qquad \forall\, \mathbf{E}^* \tag{37}$$

where $\langle \mathcal{J} \rangle$ is the global energy for the rates, defined by $\langle \mathcal{J} \rangle = \int_{\mathcal{B}} \mathcal{J} d\mathcal{B}$. Moreover, by substituting Eqs.(34) and (35) in Eq. (37) it is shown that

$$\langle \mathcal{J}(\mathcal{D}\mathbf{v}^*, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \rangle - \langle \dot{\mathbf{F}}, (\mathbf{v}^* - \mathbf{v}) \rangle \geq \langle \mathcal{J}(\mathcal{D}\mathbf{v}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \rangle \qquad \forall \mathbf{v}^* \in V \tag{38}$$

Let $\Pi(\mathbf{v})$ be defined as

$$\Pi(\mathbf{v}) = \langle \mathcal{J}(\mathcal{D}\mathbf{v}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \rangle - \langle \dot{\mathbf{F}}, \mathbf{v} \rangle = \langle \frac{1}{2} \mathbb{D}(\mathcal{D}\mathbf{v} - \dot{\mathbf{E}}_{vp}) \cdot (\mathcal{D}\mathbf{v} - \dot{\mathbf{E}}_{vp}) \rangle + \langle \frac{1}{2} \mathbf{H} \dot{\boldsymbol{\chi}} \cdot \dot{\boldsymbol{\chi}} \rangle - \langle \dot{\mathbf{F}}, \mathbf{v} \rangle \tag{39}$$

for $(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}})$ complying with the flux law (33). Thus, from (37) and from the $\Pi(\mathbf{v})$ definition (39) the Principle of Minimum Energy is stated (Feijóo and Taroco, 1980; Angelis, 2000):

Find $\mathbf{v} \in V$, such that

$$\widehat{\Pi}(\mathbf{v}) = \inf_{\mathbf{v}^* \in V} \Pi(\mathbf{v}^*) \tag{40}$$

The mixed principle is derived from (40) by applying the Legendre-Fenchel transformation

$$\langle \mathcal{J}(\dot{\mathbf{E}}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \rangle = \sup_{\dot{\mathbf{T}}^*, \dot{\mathcal{A}}} \Big[\langle \dot{\mathbf{T}}^*, \dot{\mathbf{E}} \rangle - \langle \dot{\boldsymbol{\mathcal{A}}}^*, \dot{\boldsymbol{\chi}} \rangle - \langle \mathcal{J}^c(\dot{\mathbf{T}}^*, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\mathcal{A}}}^*) \rangle \Big] \tag{41}$$

The consideration of (23) in (41), followed by the substitution of the result in (40), leads to a mixed principle, which is denoted as the Hellinger-Reissner Principle.

For $(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}})$ complying with the flux law (33), find $\mathbf{v} \in V$, $\dot{\mathbf{T}} \in W'$ such that

$$\begin{aligned} \widehat{\Pi}^{HR}(\mathbf{v}, \dot{\mathbf{T}}) = \inf_{\mathbf{v}^* \in V^0} \sup_{\dot{\mathbf{T}}^* \in W'} \Big[-\frac{1}{2} \langle \dot{\mathbf{T}}^*, \mathbb{D}^{-1} \dot{\mathbf{T}}^* \rangle + \langle \dot{\mathbf{T}}^*, \mathcal{D}\mathbf{v}^* \rangle - \langle \dot{\mathbf{T}}^*, \dot{\mathbf{E}}^{vp} \rangle + \frac{1}{2} \langle \mathbf{H} \dot{\boldsymbol{\chi}}, \dot{\boldsymbol{\chi}} \rangle - \\ \langle \dot{\mathbf{T}}^* \mathbf{n}, \mathbf{v}^* - \bar{\mathbf{v}} \rangle_{\Gamma_v} - \langle \dot{\mathbf{F}}, \mathbf{v}^* \rangle \Big] \end{aligned} \tag{42}$$

where $\langle ., . \rangle_{\Gamma_v}$ denotes the integral over Γ_v and the vector $\mathbf{n}$ is the outward unit normal to Γ.

4. Finite element models for mixed formulation

In this section a brief description of the discretization procedure is presented for the purpose of characterizing the structure of the discrete viscoplastic problem arising from the mixed principle which was presented in Section **3**. Finite element models are considered for plane stress and plane strain conditions in bodies composed by materials obeying the von Mises yield criterion and Perzyna-like viscoplastic model. For the sake of brevity, only no hardening materials are considered.

4.1. Two-dimensional models

In two-dimensional models the deformation process can be described by means of the velocity field:

$$\mathbf{v} = [\, v_x \quad v_y \,]^T \tag{43}$$

For no hardening materials, under plane stress condition the other comprised fields are

$$\dot{\mathbf{T}} = [\, \dot{T}_x \quad \dot{T}_y \ \ \sqrt{2}\, \dot{T}_{xy} \,]^T \qquad \dot{\mathbf{E}} = [\, \dot{E}_x \quad \dot{E}_y \ \ \sqrt{2}\, \dot{E}_{xy} \,]^T \qquad \dot{\mathbf{E}}_{vp} = [\, \dot{E}_{vpx} \quad \dot{E}_{vpy} \quad \sqrt{2} \dot{E}_{vpxy} \,]^T \tag{44}$$

and for the plane strain are

$$\dot{\mathbf{T}} = [\, \dot{T}_x \; \dot{T}_y \; \dot{T}_z \ \ \sqrt{2}\, \dot{T}_{xy} \,]^T \qquad \dot{\mathbf{E}} = [\, \dot{E}_x \; \dot{E}_y \; 0 \ \ \sqrt{2}\, \dot{E}_{xy} \,]^T \qquad \dot{\mathbf{E}}_{vp} = [\, \dot{E}_{vpx} \; \dot{E}_{vpy} \; \dot{E}_{vpz} \; \sqrt{2} \dot{E}_{vpxy} \,]^T \tag{45}$$

where $\dot{\mathbf{T}}$, $\dot{\mathbf{E}}$ and $\dot{\mathbf{E}}_{vp}$ are vectors which represent velocity, stress rate, total strain rate and viscoplastic strain rate fields, respectively.

Because of the vector representation of the tensorial fields the deformation operators for plane stress and plane strain states are set as

$$\mathcal{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{\sqrt{2}} \frac{\partial}{\partial y} & \frac{1}{\sqrt{2}} \frac{\partial}{\partial x} \end{bmatrix} \qquad \text{and} \qquad \mathcal{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \\ \frac{1}{\sqrt{2}} \frac{\partial}{\partial y} & \frac{1}{\sqrt{2}} \frac{\partial}{\partial x} \end{bmatrix} \tag{46}$$

In the notation of these two-dimensional models, the Von Mises yield function $f(\mathbf{T})$ is written as

$$f(\mathbf{T}) = \sqrt{\frac{3}{2}} \|\mathbf{S}\| - \sigma_Y \qquad \|\mathbf{S}\| = \sqrt{\frac{1}{2}\,\mathbb{C}\,\mathbf{T}\cdot\mathbf{T}} \tag{47}$$

where σ_Y is the material yield limit in pure traction and $\mathbf{S}$ is the deviatoric part of the tensor $\mathbf{T}$.

For plane stress and plane strain state the matrix $\mathbb{C}$ is set, respectively, as

$$\mathbb{C} = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbb{C} = \begin{bmatrix} 4/3 & -2/3 & -2/3 & 0 \\ -2/3 & 4/3 & -2/3 & 0 \\ -2/3 & -2/3 & 4/3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{48}$$

Since for Mises criterion the yield function is regular the flow rule can be expressed as in Eq (32). If, additionally, an exponential law for the Perzyna model is adopted(Perzyna, 1998; Angelis, 2000; Alfano et al., 2001), the evolution relation (32) can be written as

$$\dot{\mathbf{E}}_{vp} = \dot{\lambda}(\mathbf{T})\,\nabla_{\mathbf{T}} f(\mathbf{T}) \tag{49}$$

where

$$\dot{\lambda}(\mathbf{T}) = \eta \left(\frac{f^+(\mathbf{T})}{\sigma_Y} \right)^n \qquad \text{and} \qquad \nabla_{\mathbf{T}} f(\mathbf{T}) = \frac{1}{2}\sqrt{\frac{3}{2}}\,\frac{\mathbb{C}\,\mathbf{T}}{\|\mathbf{S}\|} \tag{50}$$

in which n is a material property.

4.2. Mixed discretization

Here a general procedure for the discretization of the mixed formulation (42) is discussed and some the particular features of proposed mixed triangular are emphasized. A curved triangular mixed element, denoted V2T1, is proposed(Borges et al., 1995; Costa and Borges, 2002), having six nodes intended for the C^0-quadratic interpolation of geometry and velocities and three nodes, at vertices, for the discontinuous linear interpolation of viscoplastic strain rates, stress rates and stresses.

Hereinafter, the following notation is adopted: a superimposed hat is used to distinguish variables or parameters of the continuum model from their discrete counterparts.

The mixed formulation assumes independent interpolations for the stress rates and the velocities. Therefore, for each element $\mathcal{T}^i$ in a triangulation $\mathcal{T}$ over the domain $\mathcal{B}$, the interpolations for velocities and stress rates are defined as

$$\widehat{\mathbf{v}}(x) = \mathbf{N}_v(x)\,\mathbf{v}^i \quad , \quad \widehat{\dot{\mathbf{T}}}(x) = \mathbf{N}_T(x)\,\dot{\mathbf{T}}^i \tag{51}$$

where the vectors $\mathbf{v}^i$, $\dot{\mathbf{T}}^i$ are the interpolation parameters for the element i. In this case $\mathbf{v}^i \in I\!R^{12}$ and $\mathbf{T}^i \in I\!R^{3\widehat{q}}$, with $\widehat{q} = 3$ for plane stress and $\widehat{q} = 4$ for plane strain. The functions $\mathbf{N}_v(x)$ and $\mathbf{N}_T(x)$ are, respectively, the matrices of quadratic and linear shape functions. The same way, viscoplastic strain rate is defined as

$$\widehat{\dot{\mathbf{E}}}_{vp}(x) = \mathbf{N}_T(x)\,\dot{\mathbf{E}}^i_{vp} \tag{52}$$

with the parameters $\dot{\mathbf{E}}^i_{vp}$ determined by considering a discrete form for the flow rule (49). The adopted interpolation for viscoplastic strain rate is chosen equal to stress because they are closed linked by the flow rule.

A discrete version for the flow rule is obtained by the collocation method, that is, a set of points are chosen in each element to enforce this rule. The points selection is addressed by the yield function behaviour. It is because the convexity of the yield function $f(\widehat{\mathbf{T}}(x))$ and the piecewise linear interpolation assumed for the stress field $\widehat{\mathbf{T}}(x)$ assure that the vertices of the triangles are the points in which the Mises yield function may be maximum. Therefore, the vertices are the natural chosen points to impose the flow rule. As a consequence, the vector $\dot{\mathbf{E}}^i_{vp}$ is assembled from three disjoint vectors $\dot{\mathbf{E}}^{ik}_{vp}$, which represents the viscoplastic strain rate at each vertex, and are determined by

$$\dot{\mathbf{E}}^{ik}_{vp} = \dot{\lambda}(\widehat{\mathbf{T}}(x^k))\,\nabla_{\mathbf{T}} f(\widehat{\mathbf{T}}(x^k)) = \dot{\lambda}(\mathbf{T}^{ik})\,\nabla_{\mathbf{T}} f(\mathbf{T}^{ik}) \qquad k = 1, 2, 3 \tag{53}$$

where $\mathbf{T}^{ik}$ represents the stress parameters at each vertex. Notice that because one can regard inter-element stress discontinuities and the coordinates x^k as coinciding with the vertices coordinates, the vector $\dot{\mathbf{E}}^{ik}_{vp} \in I\!R^{\hat{q}}$ is only dependent on a separate set, $\mathbf{T}^{ik}$, elementary vector $\mathbf{T}^i$ components.

Finally, the substitution of the assumed interpolations (51) and (52) in the continuum mixed principle (42) leads to its discrete version.

Find $\mathbf{v} \in \mathbb{R}^N$ e $\dot{\mathbf{T}} \in \mathbb{R}^q$ such that

$$\Pi^{HR}(\mathbf{v}, \dot{\mathbf{T}}) = \min_{\mathbf{v}^* \in \mathbb{R}^N} \max_{\dot{\mathbf{T}}^* \in \mathbb{R}^q} \Big[-\frac{1}{2} \mathbb{D}^{-1} \dot{\mathbf{T}}^* \cdot \dot{\mathbf{T}}^* + \dot{\mathbf{T}}^* \cdot \mathbf{B}\, \mathbf{v}^* - \dot{\mathbf{F}} \cdot \mathbf{v}^* + \dot{\mathbf{T}}^* \cdot \mathbf{B}\, \bar{\mathbf{v}} - \dot{\mathbf{T}}^* \cdot \mathbf{M}\, \dot{\mathbf{E}}_{vp} \Big] \tag{54}$$

where N is the number of degrees of freedom in velocities, assuming that all rigid motions are ruled out by prescribed kinematic constraints. Additionally, the continuity for velocities and the inter-element discontinuity for stress rates and viscoplastic strain rates are imposed by properly collecting the element vectors $\mathbf{v}^i$, $\dot{\mathbf{T}}^i$ and $\dot{\mathbf{E}}_{vp}^i$ in global vectors $\mathbf{v}$,$\dot{\mathbf{T}}$ and $\dot{\mathbf{E}}_{vp}$. Again, because of inter-element stress rate discontinuity, the rate parameters, vectores $\dot{\mathbf{T}}$ and $\dot{\mathbf{E}}_{vp}$, are made up of disjoint sets corresponding to each element. Consequently, $q = 3nel\ \hat{q}$, where nel is the total number of elements in the mesh.

The matrices $\mathbb{D}^{-1}$, $\mathbf{B}$, $\mathbf{M}$ and the vector $\dot{\mathbf{F}}$ are assembled from elementary contributions of

$$\mathbb{D}^{-1^i} = \int_{\mathcal{T}^i} \mathbf{N}_T^T\, \widehat{\mathbb{D}}^{-1}\, \mathbf{N}_T\, d\mathcal{T} \qquad\qquad \mathbf{B}^i = \int_{\mathcal{T}^i} \mathbf{N}_T^T\, \mathcal{D}\, \mathbf{N}_v\, d\mathcal{T} \tag{55}$$

$$\mathbf{M}^i = \int_{\mathcal{T}^i} \mathbf{N}_T^T\, \mathbf{N}_T\, d\mathcal{T} \qquad\qquad \dot{\mathbf{F}}^i = \int_{\mathcal{T}^i} \mathbf{N}_v^T\, \dot{\mathbf{b}}\, d\mathcal{T} + \int_{\Gamma_\tau^i} \mathbf{N}_v^T\, \dot{\mathbf{a}}\, d\Gamma_\tau \tag{56}$$

with $\widehat{\mathbb{D}}^{-1}$ given by (17).

The matrices $\mathbb{D}^{-1^i}$ and $\mathbf{M}^i$ are written as

$$\mathbb{D}^{-1^i} = \begin{bmatrix} \mathbf{A}_{11}^{-1}\widehat{\mathbb{D}}^{-1} & \mathbf{A}_{12}^{-1}\widehat{\mathbb{D}}^{-1} & \mathbf{A}_{13}^{-1}\widehat{\mathbb{D}}^{-1} \\ \mathbf{A}_{12}^{-1}\widehat{\mathbb{D}}^{-1} & \mathbf{A}_{22}^{-1}\widehat{\mathbb{D}}^{-1} & \mathbf{A}_{23}^{-1}\widehat{\mathbb{D}}^{-1} \\ \mathbf{A}_{13}^{-1}\widehat{\mathbb{D}}^{-1} & \mathbf{A}_{23}^{-1}\widehat{\mathbb{D}}^{-1} & \mathbf{A}_{33}^{-1}\widehat{\mathbb{D}}^{-1} \end{bmatrix} \qquad \mathbf{M}^i = \begin{bmatrix} \mathbf{A}_{11}^{-1}\mathbf{I} & \mathbf{A}_{12}^{-1}\mathbf{I} & \mathbf{A}_{13}^{-1}\mathbf{I} \\ \mathbf{A}_{12}^{-1}\mathbf{I} & \mathbf{A}_{22}^{-1}\mathbf{I} & \mathbf{A}_{23}^{-1}\mathbf{I} \\ \mathbf{A}_{13}^{-1}\mathbf{I} & \mathbf{A}_{23}^{-1}\mathbf{I} & \mathbf{A}_{33}^{-1}\mathbf{I} \end{bmatrix} \tag{57}$$

where $\mathbf{I}$ is the identity matrix $[3\mathrm{x}3]$ and $A_{ij}^{-1} = \int_{\mathcal{T}^e} h_i h_j\ d\mathcal{T}\ (i, j = 1, 3)$, with h_i being the lagrangean linear shape functions. The structure of these matrices and the elementary uncoupling of stress rate and viscoplastic strain rate parameters have important consequences on the computational feasibility of the discrete algorithm developed to solve this problem.

For $\dot{\mathbf{F}} \in I\!R^N$, $\mathbf{v} = \bar{\mathbf{v}} \in \Gamma_v$, let $\dot{\mathbf{E}}_{vp}$ related to the dual sate variable $\mathbf{T}$ through (53), it is easy to show that the $min - max$ principle, Eq. (54), is equivalent to the solution of the following system:

Find $\dot{\mathbf{T}} \in I\!R^q$ and $\mathbf{v} \in I\!R^N$, such that:

$$\begin{aligned} \mathbb{D}^{-1}\, \dot{\mathbf{T}} - \mathbf{B}\, \mathbf{v} - \mathbf{B}\, \bar{\mathbf{v}} + \mathbf{M}\, \dot{\mathbf{E}}^{\mathrm{vp}} &= \mathbf{0} \\ \mathbf{B}^T \dot{\mathbf{T}} - \dot{\mathbf{F}} &= \mathbf{0} \end{aligned} \tag{58}$$

The system above can be seen as a discrete version of continuum system (34-36) for no hardening materials applications. The algorithm solution for the discrete problem is based on Newton-Raphson formula for the global iteration (58) and an one-step Euler scheme for integration of viscoplastic constitutive equations (53) (Sluys, 1998).

5. Numerical application

The formulation is tested in a simple uniaxial stress problem, simulated by a two-dimensional plane stress model. It considers a bar with transversal section $\mathbf{A}$ composed by a material that has yield limite σ_Y and Young modulus $E = 1000\sigma_Y$. The load program, shown in Fig. 1, considers a maximum load equal to $2\mathbf{P}/\mathbf{A}\sigma_Y$. In Figure 2 the stress versus strain graphs is presented by having two different viscosity coefficients in a linear Perzyna model. The results are compared with analytical solutions.

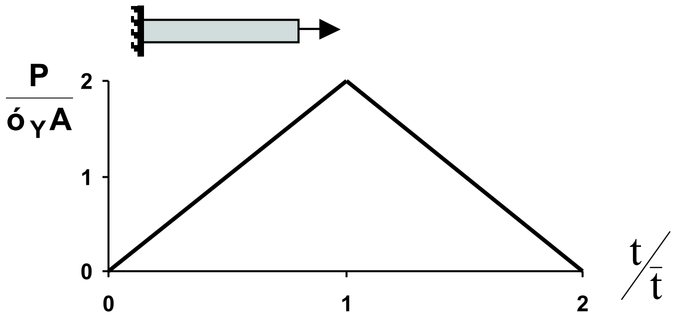

Figure 1: Uniaxial Test - Load Program

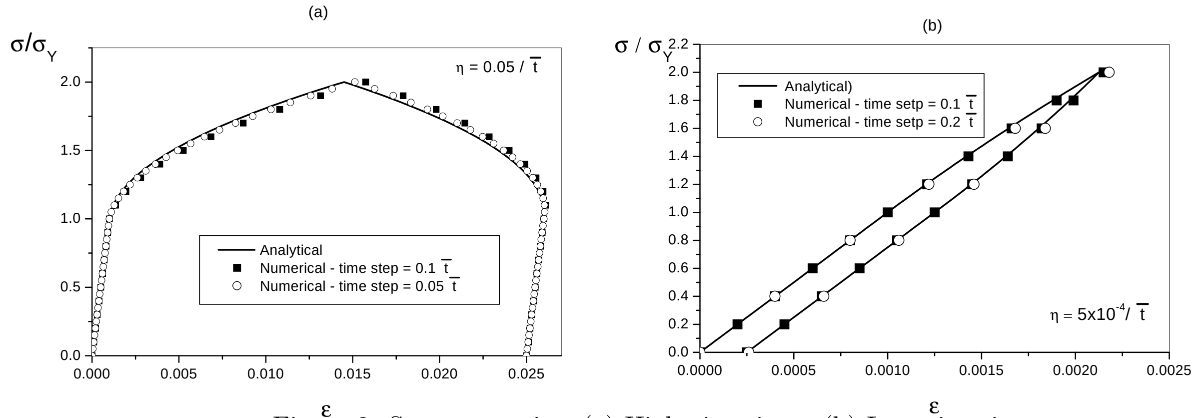

Figure 2: Stress x strain - (a) High viscosity (b) Low viscosity

6. Conclusions

A Hellinger-Reisssner variational formulation to deal with elasto/viscoplasticity problems was proposed. This mixed (*min-max*)-principle is written as function of stress rates and velocities. The viscoplastic strain rate is obtained from the flow rule and appears in the functional only as a parameter.

A mixed discretization process, based on a triangular mixed element is proposed. The element holds C^0-quadratic interpolation for geometry and velocities and a piecewise linear interpolation for viscoplastic strain rates, stress rates and stresses.

The results in a preliminary uniaxial test indicate the viability of the presented mixed methodology. More advanced applications need to be carried out in order to consolidate the formulation as an effective procedure for elasto/viscoplasticity. These advanced model must include more complex geometry, hardening or softening materials and other cycle load programs.

Such advanced features are now being preformed and will be subject of a future report.

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8. Copyright Notice

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