POINT COLLOCATION METHOD IN PLASTICITY: AXISYMMETRIC CASE

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Abstract. Point collocation method for plasticity analysis of axisymmetric bodies is developed. Numerical solution of thick cylinder subjected to internal pressure is compared with analytical solutions and FEM solutions. Excellent agreement has been found which suggests that PCM may be a valuable mean for solving problems in plasticity.

Keywords: point collocation, meshless, plasticity, axisymmetry

1. Introduction

Point collocation method is an effective technique for the solution of partial differential equations. It is also a really truly meshless method. This has attracted a number of researchers working in the computational mechanics. Of particularly interest, by using PCM, Aluru (2000), solved problems of linear elasticity and potential problems. Oñate *et al* (2001), developed a stabilized point collocation method in the linear elasticity. In this paper, we investigate the point collocation method in plasticity. This is probably a new topic because it seems to us that no related work has ever been published. The fact can not be explained by eventual lack of practical interest. Image that if one can use PCM one day for solving all problems that we have dealt with FEM so far, why would we waste our patience and time to construct meshes? FEM would have become, by that time, a historical hug monster such as Dinosaur. There should be a reason for the ignorance of PCM by researchers in elastoplastic analysis. Maybe it is due to the well known feature of PCM: the numerical instability.

In this paper, we will show a successive study of elasoplatic problem by using PCM. The moving least square approximation will be described first, as it is the starting point of any meshless methods. We'll then demonstrate that MLS function should be computed in the local coordinates to avoid losing significant digits. Authors in the committee of meshless methods have seemed so far omit this "detail". Formulation of PCM in plasticity for the case of axisymmetric problem is developed in detail. Numerical results are given which show excellent agreements with analytical or FEM solutions and hence highlight the potential of PCM in plasticity analysis.

2. Moving Least Square Approximation

With help of the MLS approximation, the displacement at point x, denoted by u(x), is related to so-called fictive displacements, $\hat{u}(x_i)$, defined at nodes scattered around x by the following relation:

$$u(x) = \sum_{i=1}^{n} \Phi_i(x)\hat{u}_i = \Phi \cdot \hat{u} \tag{1}$$

in which

$$\Phi_i(x) = \sum_{j=1}^{m} p_j(x)(A^{-1}(x)B(x))_{ji} \tag{2}$$

Taking the monomial basis of order 2 as p:

$$p = \begin{vmatrix} 1 & x & x^2 \end{vmatrix} \tag{3}$$

and the weight function as:

$$w(x - x_i) = w(r) = 1 - 6r^2 + 8r^3 - 3r^4 \tag{4}$$

the coefficients A(x) and B(x) may be computed by (see Dolbow *et al* (1998), for detail)

$$A(x)=\sum_{i=1}^{n} w(x-x_i)p(x_i)p^T(x_i) = w(x-x_i)\begin{Bmatrix}1\\ x_i\\ x_i^2\end{Bmatrix}\begin{Bmatrix}1 & x_i & x_i^2\end{Bmatrix}$$

$$= w(x-x_1)\begin{bmatrix}1 & x_1 & x_1^2\\ x_1 & x_1^2 & x_1^3\\ x_1^2 & x_1^3 & x_1^4\end{bmatrix} + w(x-x_2)\begin{bmatrix}1 & x_2 & x_2^2\\ x_2 & x_2^2 & x_2^3\\ x_2^2 & x_2^3 & x_2^4\end{bmatrix} + \cdots + w(x-x_n)\begin{bmatrix}1 & x_n & x_n^2\\ x_n & x_n^2 & x_n^3\\ x_n^2 & x_n^3 & x_n^4\end{bmatrix} \quad (5)$$

$$B(x) = w(x-x_i)p(x_i) = \begin{bmatrix} w(x-x_1)p(x_1), & w(x-x_2)p(x_2), & \cdots & w(x-x_n)p(x_n)\end{bmatrix}$$

$$= \begin{bmatrix} w(x-x_1)\begin{Bmatrix}1\\ x_1\\ x_1^2\end{Bmatrix} & w(x-x_2)\begin{Bmatrix}1\\ x_2\\ x_2^2\end{Bmatrix} & \cdots & w(x-x_n)\begin{Bmatrix}1\\ x_n\\ x_n^2\end{Bmatrix}\end{bmatrix} \quad (6)$$

The maximum value of Φ is reached at point x but is not equal to unity. The consequence is that the nodal value of $\hat{u}(x)$ at point x is not exactly equal to u(x).

In MLS approximation, the base p (quadratic) has been defined, in most literatures on meshless methods, in the global coordinates as shown in the above equations. For a typical value of x, say 100, the order of magnitude in number of matrices involved in MLS approximation are given in Table 1.

Table 1. Orders of magnitude in number of matrices involved in MLS approximation

Vector or matrix	Order of numbers
p	$1\sim10^4$
p^Tp	$1\sim10^8$
A	$1\sim10^8$
B	$1\sim10^8$
A^{-1}	$1\sim 10^{12}$
dA^{-1}/dx	$1\sim 10^{21}$
d^2A^{-1}/dx^2	$1\sim 10^{24}$
dA/dx	$1\sim 10^{15}$

It would not be surprised that the computation with quantities of such huge differences in orders will result in serious errors due to lose of significant digits. This may happen even the computation is performed with the double precision (14 significant digits) number. The remedy that we suggest is to work in the local coordinates defined at point x as depicted in Fig. (1).

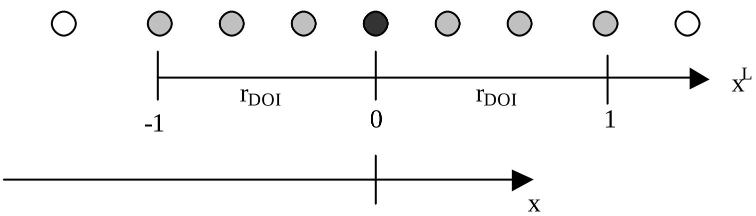


Figure 1. Local coordinates in which MLS functions are computed

$$x_i^L = \frac{x_i - x_0}{r_{DOI}} \quad (7)$$

$$x_i = r_{DOI}x_i^L + x_0 \quad (8)$$

$$\Phi(x) = \Phi(x^L) \quad (9)$$

$$\frac{d\Phi}{dx}=\frac{d\Phi}{dx^L}\frac{dx^L}{dx}=\frac{1}{r_{DOI}}\frac{d\Phi}{dx^L} \tag{10}$$

$$\frac{d^2\Phi}{dx^2}=\frac{d}{dx}\left(\frac{1}{r_{DOI}}\frac{d\Phi}{dx^L}\right)=\frac{1}{r_{DOI}^2}\frac{d^2\Phi}{d\left(x^L\right)^2} \tag{11}$$

3. Formulation of PCM in Plasticity Analysis: axisymmetric case

A point collocation method consists in satisfying the governing partial differential equation at each of the nodes inside the domain of interest. If a Dirichlet or a Neumann boundary condition is imposed on a boundary node, than an equation that satisfies the boundary condition is developed for the boundary node instead of satisfying the governing partial differential equation.

PCM may be considered as a special case of the finite difference methods (FDM). But in contrary to most FDM, PCM dose not need any knowledge on the nodal distribution rule such as the connectivity between nodes. This is the most distinguish advantage of PCM.

Denote N_d to be number of nodes carrying a Dirichlet boundary condition, N_n to be the number of nodes carrying a Neumann boundary condition and N_r to be the nodes interior in the domain. The total number of nodes covering the whole domain equals $N_d + N_n + N_r$.

To illustrate the idea, let us consider the problem of a thick cylinder subjected to an internal pressure on its inner surface. It is considered as an axisymmetric problem in this study. The extension to general cases may be performed under the similar light.

Within an axisymmetric body and by using MLS functions, the radial deformation is computed with

$$\varepsilon_r=\frac{du}{dr}=\sum_{i=1}^{m}\frac{d\Phi_i}{dr}\hat{u}_i=B\hat{u} \tag{12}$$

and the circumferential deformation with

$$\varepsilon_\theta=\frac{u}{r}=\frac{1}{r}\sum_{i=1}^{m}\Phi_i\hat{u}_i=\frac{\Phi}{r}\hat{u} \tag{13}$$

All the other components of deformation tensor as well as all shear stresses are zero:

$$\varepsilon_z=0\Rightarrow w=cst,\frac{\partial\sigma_\theta}{\partial z}=0 \tag{14}$$

$$\gamma_{r\theta}=\gamma_{\theta z}=\gamma_{rz}=0\Rightarrow\tau_{r\theta}=\tau_{\theta z}=\tau_{rz}=0 \tag{15}$$

The increment of tensor of stress is related to that of deformation by the following relationship:

$$\dot{\sigma}=\begin{Bmatrix}\dot{\sigma}_r\\ \dot{\sigma}_\theta\end{Bmatrix}=D^{ep}\dot{\varepsilon}=\begin{bmatrix}D^{ep}_{rr} & D^{ep}_{r\theta}\\ D^{ep}_{\theta r} & D^{ep}_{\theta\theta}\end{bmatrix}\begin{Bmatrix}\dot{\varepsilon}_r\\ \dot{\varepsilon}_\theta\end{Bmatrix} \tag{16}$$

in which D^{ep} stands for the tangential elasto-platic constitutive matrix.

The equilibrium equation may be written in the following form in omitting body forces:

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=0 \tag{17}$$

In the point collocation method, this differential equation is enhanced directly at the collocation points. By this nature, PCM is considered as a strong method. It is necessary to compute the derivative of stress σ_r with reference to r. To do this, we have introduced a MLS function ψ in such a way that stress at a computational point is approximated by the product of ψ and stress at nodal points around the computational point.

$$\sigma=\psi\sigma^N \tag{18}$$

We can then apply the Newton-Raphson algorithm to linearize the equilibrium equation. Let us define a functional f_1:

$$f_1 = \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{19}$$

Application of the Newton-Raphson algorithm to the functional f_1 leads to

$$\frac{\partial f_1}{\partial \hat{u}} \Delta \hat{u}^i = -f_1\left(\hat{u}^{i-1}\right) \tag{20}$$

in which $\hat{u}^{i-1}$ stands for the fictive nodal displacements resulted from the (i-1)th iteration and $\Delta\hat{u}^i$ the increment of the fictive nodal displacements to be computed in the ith iteration.

Substituting f_1 with equation (19) and σ with equation (18) in the derivative term, we obtain

$$\begin{aligned}
\frac{\partial f_1}{\partial \hat{u}} &= \frac{\partial\left(\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r}\right)}{\partial \hat{u}} = \frac{\partial}{\partial \hat{u}}\left(\frac{d\left(\psi\sigma_r^N\right)}{dr}\right) + \frac{1}{r}\left(\frac{\partial\left(\psi\sigma_r^N\right)}{\partial \hat{u}} - \frac{\partial\left(\psi\sigma_\theta^N\right)}{\partial \hat{u}}\right) \\
&= \psi' \frac{\partial \sigma_r^N}{\partial \hat{u}} + \frac{\psi}{r}\left(\frac{\partial \sigma_r^N}{\partial \hat{u}} - \frac{\partial \sigma_\theta^N}{\partial \hat{u}}\right) \\
&= \psi' \frac{\partial \sigma_r^N}{\partial \hat{u}} + \frac{\psi}{r}\left(\frac{\partial \sigma_r^N}{\partial \hat{u}} - \frac{\partial \sigma_\theta^N}{\partial \hat{u}}\right) \\
&= \psi'\left(\frac{\partial \sigma_r^N}{\partial \varepsilon_r}\frac{\partial \varepsilon_r}{\partial \hat{u}} + \frac{\partial \sigma_r^N}{\partial \varepsilon_\theta}\frac{\partial \varepsilon_\theta}{\partial \hat{u}}\right) + \frac{\psi}{r}\left[\left(\frac{\partial \sigma_r^N}{\partial \varepsilon_r}\frac{\partial \varepsilon_r}{\partial \hat{u}} + \frac{\partial \sigma_r^N}{\partial \varepsilon_\theta}\frac{\partial \varepsilon_\theta}{\partial \hat{u}}\right) - \left(\frac{\partial \sigma_\theta^N}{\partial \varepsilon_r}\frac{\partial \varepsilon_r}{\partial \hat{u}} + \frac{\partial \sigma_\theta^N}{\partial \varepsilon_\theta}\frac{\partial \varepsilon_\theta}{\partial \hat{u}}\right)\right] \\
&= \psi'\left(\frac{\partial \sigma_r^N}{\partial \varepsilon_r} B + \frac{\partial \sigma_r^N}{\partial \varepsilon_\theta}\frac{\Phi}{r}\right) + \frac{\psi}{r}\left[\left(\frac{\partial \sigma_r^N}{\partial \varepsilon_r} B + \frac{\partial \sigma_r^N}{\partial \varepsilon_\theta}\frac{\Phi}{r}\right) - \left(\frac{\partial \sigma_\theta^N}{\partial \varepsilon_r} B + \frac{\partial \sigma_\theta^N}{\partial \varepsilon_\theta}\frac{\Phi}{r}\right)\right] \\
&= \psi'\left(D_{rr}^{ep} B + D_{r\theta}^{ep}\frac{\Phi}{r}\right) + \frac{\psi}{r}\left[\left(D_{rr}^{ep} B + D_{r\theta}^{ep}\frac{\Phi}{r}\right) - \left(D_{\theta r}^{ep} B + D_{\theta\theta}^{ep}\frac{\Phi}{r}\right)\right]
\end{aligned} \tag{21}$$

Substituting the above result into equation (20), we obtain the contribution of f_1 to the system of equations as shown in the following

$$\left\{\psi'\left(D_{rr}^{ep} B + D_{r\theta}^{ep}\frac{\Phi}{r}\right) + \frac{\psi}{r}\left[\left(D_{rr}^{ep} B + D_{r\theta}^{ep}\frac{\Phi}{r}\right) - \left(D_{\theta r}^{ep} B + D_{\theta\theta}^{ep}\frac{\Phi}{r}\right)\right]\right\}\Delta\hat{u}^i = -\left(\psi'\sigma_r^{i-1} + \frac{\psi}{r}\left(\sigma_r^{i-1} - \sigma_\theta^{i-1}\right)\right) \tag{22}$$

For the governing equation corresponding to BV of Neumann type, we define a functional f_2 as given by

$$f_2 = \sigma_r \cdot n_r - \bar{t}_r = 0 \tag{23}$$

Application of Newton-Raphson algorithm to f_2 leads to

$$\frac{\partial f_2}{\partial \hat{u}} \Delta \hat{u}^i = -f_2\left(\hat{u}^{i-1}\right) \tag{24}$$

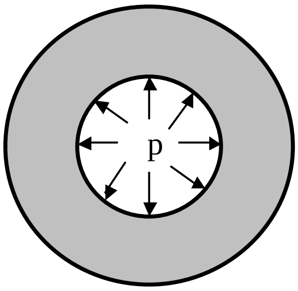
$$\frac{\partial f_2}{\partial \hat{u}} = \frac{\partial \sigma_r}{\partial \hat{u}} n_r = \left(\frac{\partial \sigma_r}{\partial \varepsilon_r}\frac{\partial \varepsilon_r}{\partial \hat{u}} + \frac{\partial \sigma_r}{\partial \varepsilon_\theta}\frac{\partial \varepsilon_\theta}{\partial \hat{u}}\right) n_r = \left(D_{rr}^{ep} B + D_{r\theta}^{ep}\frac{\Phi}{r}\right) n_r \tag{25}$$

The contribution of f_2 to the system of equations may be written as

$$\left(D^{ep}_{rr}B + D^{ep}_{r\theta}\frac{\Phi}{r}\right)n_r\Delta\hat{u}^{i} = -\left(\sigma_r^{i-1}\cdot n_r - \bar{t}_r\right) \tag{26}$$

4. Numerical Results

Consider a thick cylinder subjected to a gradually increasing internal pressure. The problem is solved with assumption of plan strain condition in the axial direction. Data held in the computation are given in the following figure.



Inner radius, a = 0.1m
External radius, b = 0.2 m
Elastic modulus, E = 2.1x10^5 MPa
Poisson's ratio, μ = 0.3
Yield stress, σ_s = 240 MPa
Von Mises yield function
Associate perfect plasticity
Internal pressure, p = 200 MPa

Figure 2. Definition of thick cylinder subjected to an internal pressure

101 uniformly distributed points are put along a radial direction as shown in the following figure. The most left point, point 1, corresponds to the inner surface of the cylinder and the most right point, point 101, corresponds to the exterior surface that is free of loading. Equation (26) is applied to both of the two points. All the other points are interior points where the linearized equilibrium equation (22) is applied.

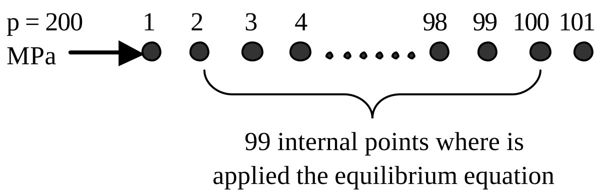


Figure 3. Meshless discretization with 101 equally distanced collocation points

In this computation, the MLS function for the displacement, Φ, and for the stress, ψ, are the same. The elastic solution under the internal pressure of the nominal value, i.e., 200 MPa, is first solved. Radial stress σ_r and hoop stress

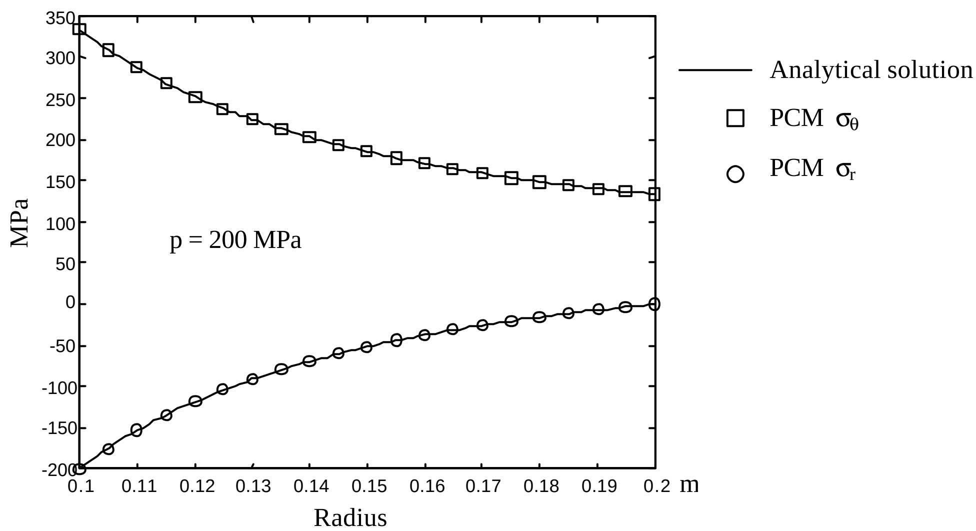


Figure 4. Elastic solution of stresses corresponding to the nominal value of pressure

σ_θ of this solution are plotted in Fig.(4). Excellent agreement with the analytical solution can be observed.

Check of the elastic solution revealed that the first plastic yielding took place at the pressure level equal to 103.75 Mpa, i.e., 51.88% of the nominal pressure. The elastic solution is then reduced by multiplying its full value with 51.88%. This solution served as the starting point for the forthcoming elastoplastic computation. Further loading is then applied with an incremental process. Initially, the step size is defined as 5% of the nominal value of pressure. Iterations are performed to retrieve the equilibrium within each step of loading. The test of convergence is based on the residual value of the right hand side of the system of equations. If the maximum number of iterations, fixed as 6 in this computation, is reached without obtaining the convergence, the step size is reduced by half. If the minimum step size is reached by this way, the computation is stopped. All the algorithms presented in this paper have been implemented in MATLAB.

It is shown in Fig. (5) the pressure against the radial displacement at point 1 that is on the inner surface of the cylinder. The solution is compared with the one obtained with a program of the finite element method. The FEM program has been developed by the author as well within MATHEMATICA. It can be observed that PCM solution is in excellent agreement with the FEM solution.

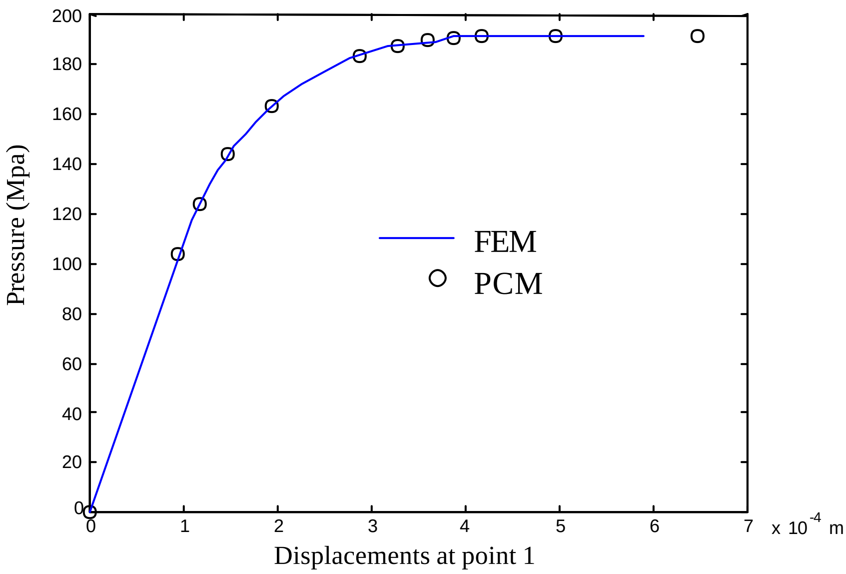


Figure 5. Elasto-plastic solution of displacements vs. pressure.

The value of pressure of last converged step is equal to 191.88 Mpa. It is well known that the "full plastic" pressure of the problem, p_p, may be estimated with the following relationship.

$$p_p = \frac{2\sigma_s}{\sqrt{3}} \ln\left(\frac{b}{a}\right) \tag{27}$$

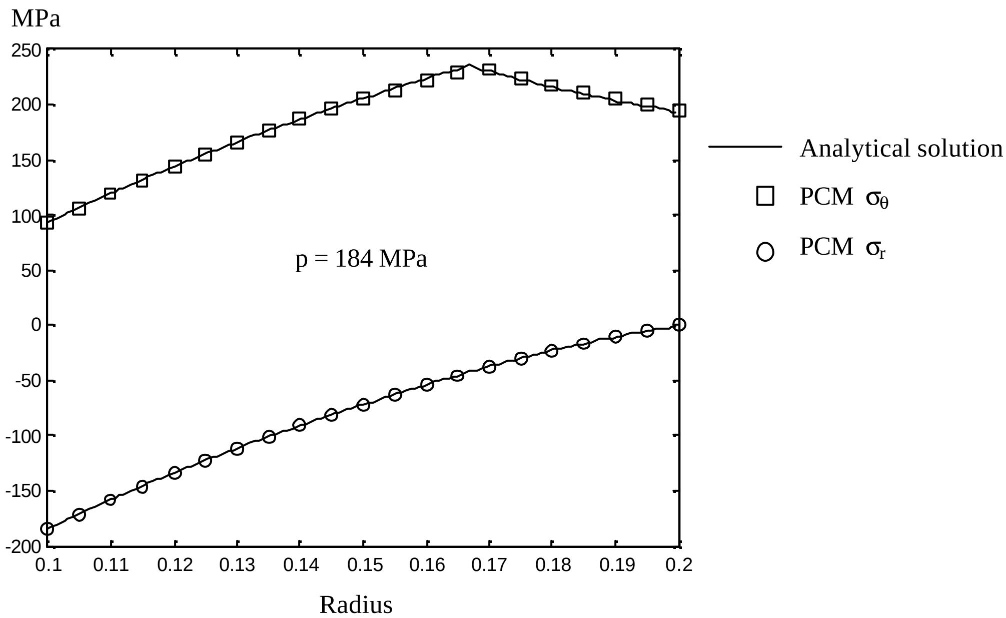
Note that the relationship has been obtained by multiplying the analytical solution corresponding to the criterion of Tresca by factor of $2/\sqrt{3}$. Hereafter, when referring to an analytical solution, we mean that the analytical solution corresponding to the criterion of Tresca in which the yield stress, σ_s, is multiplied by $2/\sqrt{3}$. The value of p_p computed with the equation (27) is 192.09 Mpa. The relative difference of the value given by PCM solution with reference to this p_p is 0.1%.

We have also checked stresses of PCM solution with the analytical solutions given in Table 2 in which the radius of the plastic zone, r_p, may be found in solving Eq. (28). For a given value of pressure being equal to 184 Mpa, r_p is equal to 0.167 m. In PCM solution, this value is found to be equal to 0.169 m. The relative difference is 1.2%. Radial and hoop stresses of both PCM and the analytical solutions are plotted in Fig. (6). Again, excellent agreement can be observed.

$$\frac{\sqrt{3}p}{2\sigma_s} = \ln\left(\frac{r_p}{a} + \frac{1}{2}\left(1 - \frac{r_p^2}{b^2}\right)\right) \tag{28}$$

Table 2 Analytical elastoplastic solution of stress of thick cylinder

Plastic zone: $a \le r \le r_p$	Elastic zone: $r_p \le r \le b$
$\sigma_r = \frac{2\sigma_s}{\sqrt{3}}\left[\ln\frac{r}{r_p} - \frac{1}{2}\left(1 - \frac{r_p^2}{b^2}\right)\right]$	$\sigma_r = \frac{\sigma_s r_p^2}{\sqrt{3}}\left[\frac{1}{b^2} - \frac{1}{r^2}\right]$
$\sigma_\theta = \frac{2\sigma_s}{\sqrt{3}}\left[\ln\frac{r}{r_p} + \frac{1}{2}\left(1 + \frac{r_p^2}{b^2}\right)\right]$	$\sigma_\theta = \frac{\sigma_s r_p^2}{\sqrt{3}}\left[\frac{1}{b^2} + \frac{1}{r^2}\right]$



In setting the tolerance of convergence to be zero, i.e., the iteration would never be considered as converged, we have checked if the iterative solution suffered eventually from the numerical instability. It has been found that after 7 iterations, the right hand side of the system of equations was reduced to be 0.00000000000000 and that it remained true for up to 50 iterations. This suggests that PCM for the solution of 1-D axisymmetric problem in plasticity would not be suffered from the numerical instability.

5. Conclusions

We have presented a point collocation method for elastoplastic analysis. From our knowledge, such a kind of study has never been published. Results of analysis of an axisymmetric cylinder with PCM are in excellent agreement with that given by using FEM and by the analytical solution. The numerical instability has not been observed in this computation. Extension of the method to 2D analysis is under the way and results will be presented in a future paper.

6. References

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