INEXACT-RESTORATION ALGORITHMS FOR BILEVEL PROGRAMMING PROBLEMS

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Abstract. *We propose the use of inexact-restoration algorithms to solve bilevel programming problems. We are specially interested in the case where the follower´s problem is an optimization problem or a variational equality or inequality problem. We discuss the application of these techniques to shape optimization problems.*

Keywords*. Bilevel problems, inexact-restoration, shape optimization.*

1. Introduction

The bilevel mathematical programming problem is characterized by an optimization problem whose constraints set is described by another optimization problem, or by a variational equality or inequality problem. This structure often appears in the mathematical formulation of many practical problems that involve an hierarchical decision, as in game theory, in engineering design and chemical design process. Besides, some classical mathematical programming problems, as min-max and bilinear optimization problems, are special cases of bilevel problems.

Due to the interaction between the two optimization problems, bilevel problems are considerably more difficult than standard optimization problems. They are usually non convex and non differentiable.

A number of authors have established optimality conditions and techniques to deal with special classes of bilevel problems. They are usually based on the reformulation of the problem (as in Clark and Westerberg, 1990 and Campêlo, 1999) and/or the use of non differentiable techniques (Shimizu et all, 1997).

In this work, we propose to solve quadratic bilevel problems using an Inexact-Restoration algorithm proposed by Martínez and Pilotta (2000). This is a method for solving classical nonlinear programming problems, that has some properties that permit to exploit the characteristics of bilevel problems.

This paper is organized as follows. In section 2 we describe the bilevel mathematical programming problems. In section 3 we comment on the Inexact-restoration algorithm. In section 4 we present our approach for solving quadratic bilevel problems by an Inexact-Restoration algorithm. In section 5, we describe some numerical experiments, including the resolution of a mechanical problem: shape optimization in contact problems. Conclusions and future work are discussed in Section 6.

2. The bilevel programming problem

The bilevel problem was first formulated in the work of Bracken and McGill (1973) and it received this designation by Candler and Norton (1977). Since 1980, several authors have studied bilevel problems intensively, specially motivated by the game theory of Stackelberg (1952), and its practical importance. In the field of applications we can cite game theory, optimal design problems in mechanical engineering, chemical process design and network design.

The bilevel problem is characterized by an optimization problem (denoted by first level or leader's problem) which has a subset of its variables constrained to be a solution of another optimization problem parameterized by their remaining variables (denoted by second level or follower's problem). It may be expressed as follows:

$$\begin{aligned} &\underset{x,y}{\text{Minimize}} \;\; F(x,y) \\ &\text{subject to} \begin{cases} G(x,y) \le 0 \\ y = \arg\underset{y}{\min} \;\; f(x,y) \\ \qquad\quad \text{s.t. } g(x,y) \le 0 \end{cases} \end{aligned} \tag{1}$$

where $x \in \Re^{nx}$, $y \in \Re^{ny}$, F, $f : \Re^{nx+ny} \to \Re, G : \Re^{nx+ny} \to \Re^{m1}, g : \Re^{nx+ny} \to \Re^{m2}$ are continuous functions and the constraints sets are compact.

Bilevel problems are classified according to the class of functions F and f and the constraints G and g. Three important classes are: the linear bilevel problems, where all the involved functions are linear; the linear-quadratic bilevel problems, where the follower's problem is a quadratic problem and the quadratic bilevel problems. If the follower's problem is replaced by a variational equality or inequality we get the Generalized Bilevel Problem.

Some classical optimization problems can be stated as special cases of bilevel problems. For example, the min-max and bilinear programs. Other important problems that are related with bilevel problems are the Stackelberg and the multi-objective optimization problems.

Bilevel problems are considerably more difficult than classical programming problems. The interaction between the leader and the follower produces a feasible set that is usually not convex, not differentiable and sometimes disconnected or empty.

Several different optimality conditions and algorithms have been proposed in the literature to deal with bilevel programs. An usual technique consists in replacing the follower's problem by its Karush-Kuhn-Tucker conditions and turning it into a classical constrained optimization problem (Clark and Westerberg, 1990 and Campêlo, 1999). In this approach some difficulties are introduced, as the complementarity conditions and the appearance of many spurious local optimizers, whose significance for the original problem is dubious. Other techniques that have been used to study the problems are based on non-differentiable approaches (Shimizu et all, 1997).

3. Inexact-restoration algotithms

An Inexact-Restoration algorithm is a method for solving nonlinear programming problems in the form:

$$\begin{aligned} &\text{Minimize } f(x) \\ &\text{subject to} \begin{cases} C(x) = 0 \\ x \in \Omega \end{cases} \end{aligned} \tag{2}$$

where $f:\Re^n \to \Re, C:\Re^n \to \Re^m$ are continuous functions, $\Omega \subset \Re^n$ is a compact set and $\nabla f(x)$ and $C'(x)$ exist and are continuous.

The method consists in an iterative process that at each iteration proceeds in two phases. In the first one, that is called restoration phase, the feasibility of the current iterate is improved. In the second, that is called minimization phase, the objective function value is reduced in an approximate feasible set.

There are different Inexact Restoration algorithms available. The one that we have studied was proposed by Martínez (2001). It uses the trust region approach and the global convergence is proved based on merit functions of the augmented Lagrangian type.

The feasible phase at this algorithm basically consists in, given an iterate x^k, $r \in (0,1)$ and $\beta > 0$, to determine a point $y^k \in \Omega$ such that:

$$\left|C(y^k)\right| \leq r\left|C(x^k)\right| \tag{3}$$

$$\left\|x^k - y^k\right\| \leq \beta\left|C(x^k)\right| \tag{4}$$

where $|.|$ denotes the Euclidean norm and $\|.\|$ denotes an arbitrary norm in $\Re^n$.

The minimization phase consists in, given the point y^k that satisfies the Eqs.(3)-(4), to determine a point $z^k \in \Pi^k = \left\{z \in \Omega \middle| C'(y^k)[z - y^k] = 0\right\}$ such that:

$$l(y^k, \lambda^k) - l(z^k, \lambda^k) >> 0 \tag{5}$$

where $l(x,\lambda) = f(x) + \langle C(x), \lambda \rangle$ is the Lagrangian function, $\Pi^k = \left\{z \in \Omega \middle| C'(y^k)[z - y^k] = 0\right\}$ is the feasible tangent linear manifold that passes through y^k and the symbol $>>$ means that the involved reduction is sufficiently large.

The details of the algorithm's steps, and the convergence results of the algorithm, are in Martínez and Pilotta (2000) and in Martínez (2001).

4. Inexact-restoration algorithm for solving quadratic bilevel programming

We have two main reasons for thinking that Inexact-Restoration algorithms are a good way to deal with bilevel problems. The first one is that in these algorithms we are free to choose the technique for restoration. It allows us to exploit characteristics of the constraints, which for bilevel problems means that it is possible to solve the follower's problem directly by any convenient available algorithm.

The second reason is that in many practical applications feasibility of the solution is more important than optimality, since the follower's problem usually describes a state equation.

In this work we apply the Inexact-restoration algorithm proposed in Martínez (2001) for solving a special class of bilevel problems. We consider the follower's problem as a strict convex quadratic problem and we suppose that there are no additional constraints for the leader's problem. It can be stated as follows:

$$
\begin{array}{l}
\underset{x,y}{\text{Minimize}} \;\; F(x,y) \\
\text{subject } to \begin{cases} L \le x \le U \\ y \;=\; \arg\underset{y}{\min}\;\; f(x,y) \equiv y^T A y + y^T b(x) + c(x) \\ \qquad\qquad \text{s.t. } Dy \;=\; \mathrm{d}(x),\;\; l \le y \le u \end{cases}
\end{array}
\tag{6}
$$

where $A \in \Re^{ny \times ny}$ is a symmetric, positive defined matrix, $D \in \Re^{m \times ny}$ is a full rank matrix, $b,c \in \Re^{ny}$, $d \in \Re^{m}$, $L,U \in \Re^{nx}$ and $l,u \in \Re^{ny}$.

The steps of the Inexact-Restoration algorithm applied to solve the bilevel problem in Eq.(6) are described in the following algorithm. It differs from the one in Martínez (2001) just in the way that we deal with the constraints (restoration and minimization phase).

4.1. Algorithm

Let us consider the Karush-Kuhn-Tucker (KKT) optimality conditions of the follower's problem and define:

$$
C(x,y,\gamma,\mu,\tau) = \begin{pmatrix} Ay + b(x) + D^T\gamma - \mu + \tau \\ Dy - d(x) \\ \mu^T (l-y) \\ \tau^T (y-u) \end{pmatrix}
$$

where (γ,μ,τ) are an estimation of the Lagrange multipliers for the constraints, associated with the point (x,y).

Given $r_k \in (0,1)$, $\beta > 0$, $\delta_{\min} > 0$, $\delta_k > \delta_{\min}$, $(x^k, y^k) \in \Omega$ and $\lambda^k \in \Re^{m+ny+2}$, the steps for obtaining the new iterate (x^{k+1}, y^{k+1}) are:

<u>Step 1:</u> Restoration phase.

Given the point x^k, compute the point $\bar{y}$, such that $(x^k, \bar{y}) = (\bar{x}, \bar{y})$ is an inexact solution of the follower's problem:

$$
\begin{array}{l}
\underset{y}{\text{Minimize}} \;\; f(x^k,y) \equiv y^T A y + y^T b(x^k) + c(x^k) \\
\text{subject } to \; \{ \; Dy \;=\; \mathrm{d}(x^k),\;\; l \le y \le u
\end{array}
\tag{7}
$$

and such that

$$
\left\| \bar{y} - y^k \right\| \le \beta \left| C\;\; (x^k, y^k, \gamma^k, \mu^k, \tau^k) \right|
\tag{8}
$$

where $(\gamma^k, \mu^k, \tau^k)$ are the estimated Lagrange multipliers associated with the point (x^k, y^k).
Inexactly solving the Eq.(7) means that the stopping criteria of the algorithm used in this step is:

$$
\left| C\;\; (x^k, \bar{y}, \bar{\gamma}, \bar{\mu}, \bar{\tau}) \right| \le r_k \left| C\;\; (x^k, y^k, \gamma^k, \mu^k, \tau^k) \right|
\tag{9}
$$

where $(\bar{\gamma}, \bar{\mu}, \bar{\tau})$ are the estimated Lagrange multipliers associated with the point $(\bar{x}, \bar{y})$.

<u>Step 2:</u> Tangent Cauchy direction.

Compute

$$
d^k_{\tan} = P_k\left[(\bar{z}, \bar{\lambda}) - \eta \nabla L(\bar{z}, \bar{\lambda}) \right] - (\bar{z}, \bar{\lambda})
$$

with $P_k(w)$ the orthogonal projection of w on $\Pi^k(\bar{z})$

where

$$
L(z,\lambda) = F(x,y) + \langle C\;\; (x,y,\gamma,\mu,\tau), \lambda \rangle,
$$

$$
\Pi^k(\bar{z}) = \left\{ z \in \Omega \subset \Re^{nx+ny+m+2ny} \;\middle|\; C'\;(\bar{x}, \bar{y}, \bar{\gamma}, \bar{\mu}, \bar{\tau}) \left[z - (\bar{x}, \bar{y}, \bar{\gamma}, \bar{\mu}, \bar{\tau}) \right] = 0 \right\}
$$

and

$$\Omega = \left\{(x, y, \gamma, \mu, \tau) \in \Re^{nx+ny+m+2ny} \;\middle|\; L \le x \le U,\;\; l \le y \le u,\;\; m \le (\gamma, \mu, \tau) \le M \right\}.$$

If $(\bar{x}, \bar{y}) = (x^k, y^k)$ and $d_{\tan}^{k} = 0$, terminate the execution of the algorithm returning (x^k, y^k) as the solution.
If $d_{\tan}^{k} = 0$, then $(x^{k+1}, y^{k+1}) = (x^k, y^k)$ and terminate the iteration.

<u>Step 3:</u> Minimization phase.
Compute the point $\bar{\bar{z}} \in \Pi^k$ such that:

$$L(\bar{\bar{z}}, \lambda^k) << L(\bar{z}, \lambda^k) \text{ and } \left\| \bar{\bar{z}} - \bar{z} \right\| \le \delta_k$$

<u>Step 4:</u> Acceptance of the new iterate.
The point $\bar{\bar{z}}$ is accepted or rejected based on an merit function (Martínez, 2001).
If it is rejected, choose $\delta_k \in [0.1\delta_k, 0.9\delta_k]$ and go to step 3.

More details of the algorithm's steps can be found in Martínez and Pilotta (2000) and Martínez (2001).

4. 2. Algorithm's convergence

According to the convergence theorem described in Martínez (2001), under suitable hypothesis, the inexact-restoration algorithm generates a sequence such that every accumulation point is feasible, the Cauchy directions tend to zero and, under a suitable condition, there is a accumulation point that satisfies the KKT condition.

In the case of the bilevel problem, we need to verify two hypothesis: the differentiablility of the functions defining the constraints (in order to compute the approximated tangent set, in the minimization phase) and the possibility to complete step 1 (feasible phase) at each iteration.

The first one is true, since in order to define the tangent set we have used the KKT conditions of the follower's problem. About the second one, the following result can be proved:

If the problem in Eq.(7) is solved by any optimization algorithm with global properties of convergence (in the sense that the algorithms stops in a finite number of steps with a point that satisfies the KKT equations or generates a sequence of iterates such that every accumulation point satisfies the KKT equation) then for a given $r \in (0,1)$*, there exist* $\beta > 0$ *and* $(\bar{x}, \bar{y})$ *that satisfy Eqs. (8)-(9).*

5. Numerical results

We have implemented the Inexact-restoration algorithm, described in section 4.1**,** in *Fortran 90*, for solving the collection of bilevel test problems proposed in Floudas et all (1999). It consists of ten linear bilevel problems and nine quadratic bilevel problems.

We obtained the best known global solution in all the test problems. However, for different initial points than the ones proposed in the related article, local solutions were found in some problems.

This tests on the algorithm encouraged us to apply it for solving larger bilevel problems. The one that we have studied is in the structural mechanical field and is described in the next subsection.

5.1. Shape optimization in contact problems

The contact problem (Kikuchi and Oden, 1988) describes the contact of a linearly elastic body $\mathrm{B} \subset \Omega \subset \Re^3$ with a rigid foundation $\Im$. The body is submitted to surface traction f and body forces b. Imposing the equilibrium conditions, after finite element discretization, the displacement is calculated solving the quadratic problem:

$$\underset{v \in \mathrm{K}}{\text{Minimize}} \;\; \frac{1}{2} a_\Omega(v, v) - l_\Omega(v) \tag{10}$$

where K is a convex set of displacements and a_Ω and l_Ω are given bilinear and linear forms, respectively, typical in contact problems (Kikuchi and Oden, 1998).

The goal in the shape optimization problem is to find the body shape Ω (in a certain admissible set Γ) so that stress distribution in the contact surface is constant.

If we consider that the shape is characterized by a variable $\aleph$, the discretized problem can be stated as follows:

$$\underset{\aleph \in \Theta_h}{\text{Minimize}} \;\; \Phi(\aleph, u_h(\aleph)) \tag{11}$$

where $u_h(\aleph)$ is the solution of the frictionless contact problem of Eq.(10).

The problem (10)-(11) can be written as the following bilevel problem:

$$\begin{aligned} &\underset{\aleph, u}{\text{Minimize}} \;\; \Phi(\aleph, u) \\ &\text{subject } to \begin{cases} u \; = \; \underset{u \,\in\, K}{\arg\min} \;\; \frac{1}{2} a_\Omega(u,u) - l_\Omega(u) \end{cases} \end{aligned} \tag{12}$$

The shape optimization problem we solved consists in determining the shape of an elastic cantilevered beam over a rigid foundation described in Kikuchi and Oden (1888). The beam is modeled as a thin body in state of plane stress and the loading and constraint configuration are illustrated in Figure 1.

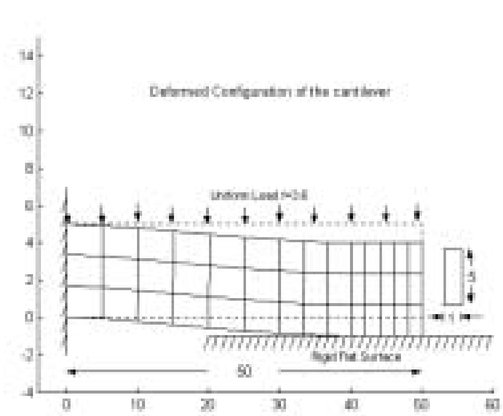

Figure 1. The contact problem

The cost function consists, as in Fancello and Feijóo (1994), in the total potential energy evaluated at the equilibrium state. The changes in shape are due to the variation of the contact boundary. In this example, the design variable that describes the boundary is its angle of inclination.

The Inexact-Restoration algorithm (in section 4.1) applied to solve Eq.(12) was able to find the solution, but its convergence was slow. The algorithm was implemented for a general problem, so no benefit is taken of any special structure of the problem. The solution is shown in Figure 2.

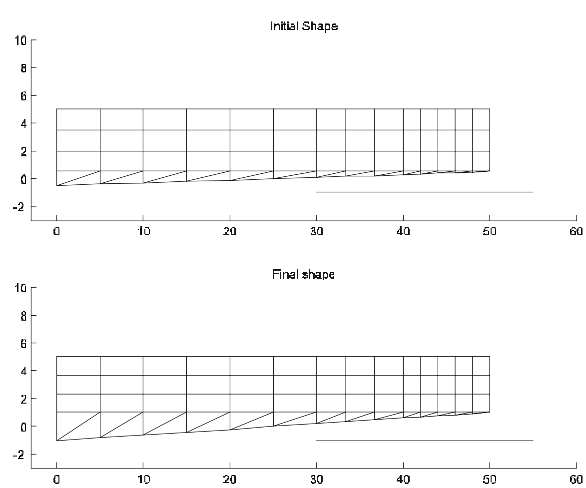

Figure 2. Solution of the shape optimization problem.

6. Conclusions and future work

We have proposed a technique for solving bilevel programming problems by an Inexact-Restoration algorithm. In despite of others techniques, it permits to solve the follower's problems directly, using any mathematical programming algorithm available.

The technique was used to solve a special class of bilevel problems, where the follower's problem is a quadratic strictly convex optimization problem. It has showed a good performance for solving the collection of test problems (Floudas et all, 1999) and was able to solve a larger problem arising in mechanical engineering.

We believe that this technique could be a good choice to solve more general problems. We are specially interested in the shape and topological optimization problems with non-linear state equations, as in the case of non-linear materials.

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