

FINITE ELEMENT EVALUATION OF FATIGUE STRENGTH

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Abstract. The goal of the present paper is to assess the performance of some multiaxial fatigue criteria when incorporated into an academic finite element code. Crossland and a new model proposed by the authors were the criteria considered in the present work. Experimental data collected from the literature are considered to validate the analysis. Results showed that the new fatigue criterion presented results signifcantly better than the Crossland criterion. Further, the incorporation of such model to a FE code gave rise to an extremelly powerful engineering design tool.

***Keywords**. multiaxial fatigue, endurance criterion, finite element.*

1. Introduction

Fatigue can be understood as the mechanical degradation and failure of the material due to repeated loading, even at levels well below its ultimate strength. The subject has been studied since the beginning of the nineteenth century, mainly after the railway accident near Versailles, France, due to the fatigue failure of the locomotive front axle. Schütz (1996) reports many airplane accidents related to the fatigue phenomenon, starting at the end of the Second World War. Probably the most impacting case of airplane fatigue failure occurred with the Comet, the first commercial jet airplane in the world. Comet planes disintegrated in the air in may 1953, in January and in may 1954, leading to an intensive program of investigations of the causes, which led to the conclusion that the failures were caused by fatigue due to pressurization-depressurization cycles.

It has been estimated (Milne, 1994, Ree et al., 1983) that the fracture of engineering components costs the economy of developed countries 4% of the gross national product (GNP). Further, around 90% of engineering failures are due to fatigue and most of these are caused by poor design methodologies (Akrache & Lu, 1999). Structural or mechanical components are usually subjected to complex service loading, which produces a multiaxial state of stress that must be, in a number of cases, sustained for a high number of fatigue cycles (HCF). Under such conditions the use of stress based models is appropriate although it is still possible to use strain-stress based models such as the critical plane criteria proposed by Socie (1987). However, this class of criteria are usually more expensive computationally than stress based criteria as they require the search for a critical material plane.

In this paper we assess the performance of two multiaxial fatigue endurance criteria when incorporated into an academic finite element code. Experimental results collected from the literature are considered to validate the analysis. The paper is organized as follows: the multiaxial fatigue endurance criteria of Crossland and the one proposed by the authors are described in detail in section 2. The criteria are assessed in section 3, while the computational issues are addressed in section 4. Then, section 5 is devoted to the assessment of the incorporation of the fatigue criteria into a finite element code. Final remarks are presented in section 6.

2. Fatigue criteria

The S-N curve, first proposed by Wöhler (1860), together with Goodman or Gerber diagrams, give us simple but efficient tools for predicting fatigue strength under conditions of essentially uniaxial loadings. On the other hand, when the material is subjected to multiaxial loadings, the extension of these techniques is not a straightforward task and has been the subject of intensive research.

Many factors contribute to the degradation of metallic materials when subjected to cycling loadings, including cyclic plastic deformations (at mesoscopic or macroscopic levels) and tensile hydrostatic stress states. Cyclic plastic deformations lead to formation of dislocations, which form a lamellar structure known as Persistent Slip Bands (PSBs), which are believed to be the first stage of crack nucleation. Thus, shear stress (which is the driving force of plastic deformations) is directly involved in the degradation process under conditions of cyclic loadings. In this setting, fatigue can be classified as follows: (i) if plasticity can be detected at macroscopic level, then we have the so called low cycle fatigue; (ii) on the other hand, if the macroscopic behavior is essentially elastic but, at mesoscopic (crystal) level, plastic deformations can still be detected, then fatigue failure usually is expected only after a very large number of cycles (tipically 10^6 cycles) and thus it is known as high cycle fatigue; (iii) finally, if elastic behavior is observed even at mesoscopic level (eventually after a number of plastic cycles, but then attaining elastic shakedown), then fatigue failure is not expected to occur. The highest level of the stress amplitude in the latter case defines the so called fatigue endurance of the material.

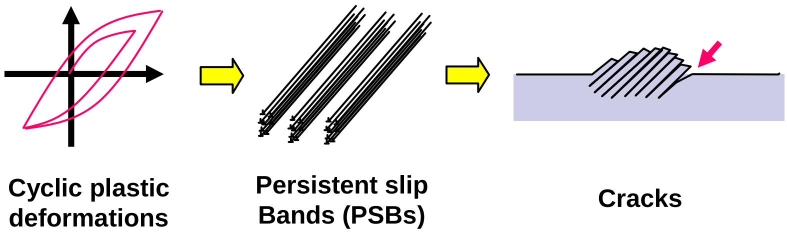


Figure 1. Nucleation of fatigue cracks due to cyclic plastic deformations.

Further, tensile normal stresses acting on the surfaces of embryo-cracks contribute to the fatigue mechanical degradation of the material as has been observed by experimental work (Sines & Ohgi, 1981). Its influence has been taken into account by many authors through an average of the normal stress acting upon all the planes passing through the material point. As remarked by Papadopoulos (1997) such average is equal to the hydrostatic stress.

Based on the aforementioned arguments, many multi-axial fatigue endurance criteria can be expressed as:

$$f(\tau) + g(\sigma) \leq \lambda \,, \tag{1}$$

where $f(\tau)$ is a function of the shear stress history, $g(\sigma)$ is a function of the normal stress history, while λ is a material parameter.

2.1. The Crossland criterion

The criterion proposed by Crossland (1956), for instance, can be written as:

$$\sqrt{J_{2,a}} + \kappa \, p_{\max} \leq \lambda \,, \tag{2}$$

where $\sqrt{J_{2,a}}$ is the radius of the minimum hypersphere containing the deviatoric stress path, as illustrated in Fig. 2 and $p_{\max}$ is the maximum value of the hydrostatic stress observed along the stress history, while κ and λ are material paramenters. Interesting features of this criterion include: (i) the concept of stress amplitude is generalized to multiaxial stress histories, (ii) mean shear stresses do not influence the fatigue strength and (iii) tensile normal stresses contribute to fatigue damage. On the other hand, this criterion is not able to make distinction between proportional and nonproportional stress histories circumscribed by the same hypersphere and under the same value of the maximum hydrostatic stress.

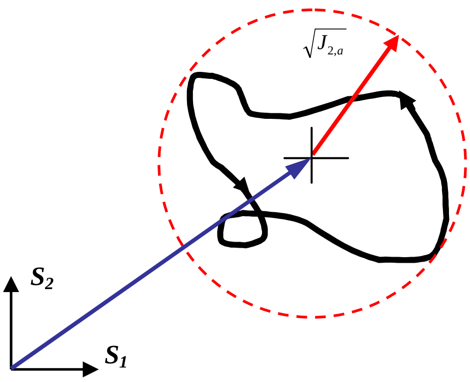


Figure 2. Minimum hypersphere circumscribing the stress path in the deviatoric stress space.

2.2. A criterion based on a new measure of shear solicitation to fatigue

More recently, a number of fatigue endurance criteria capable of making distinction between proportional and nonproportional loadings have been proposed. Deperrois (1991), for instance, states that the shear solicitation to fatigue can be expressed as:

$$f(\tau) = \frac{\sqrt{2}}{2}\sqrt{\sum_{i=1}^{5}\lambda_i^2} \ , \tag{3}$$

where $\lambda_i \ i = 1,\ldots,5$ are computed as follows: first, the longest chord D_5 between two distinct points of the stress path in the deviatoric space is determined; next, the stress path is projected into a subspace orthogonal to such chord; a new longest chord is computed in this subspace, and the process is repeated successively for the remaining dimensions. This criterion has been severely criticized due to the lack of uniqueness of the chords in some topological situations and consequent ill posedness of the chord search problem. Nevertheless, it represents a very appealing model due to its remarkably good quality of the associated prediction of fatigue strength whenever the longest chord in each direction is well determined. Bin Li et al. (2000) have proposed a modification of the model by substituting the longest chords – as measure of stress amplitudes – by the lengths of the semi-axes $\lambda_i \ i = 1,\ldots,5$ of the ellipsoid circumscribing the stress path (always in the deviatoric space), as illustrated in Fig. 3.

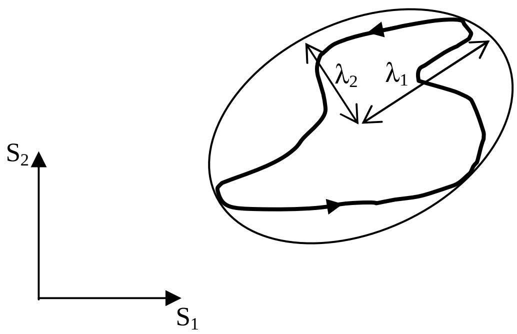


Figure 3. Semi-axes of the elliposoid circumscribing the stress path as stress amplitudes within the multi-axial setting.

An alternative definition of the shear solicitation to fatigue is proposed by Mamiya & Araújo (2002). The basic argument is that, following the ideas proposed in their essence by Deperrois, the ellipsoid circumscribing the stress history in the deviatoric stress space (and its geometrical quantities) gives a good characterization of the shear solicitation to fatigue. The difficulty in applying these criteria is related to the elaborate and time consuming numerical calculations involved in the determination of the semi-axes $\lambda_i, i = 1,\ldots,m$ of the circumscribing ellipsoid.

In this sense, Mamiya & Araújo (2002) proved that it is possible to characterize the shear solicitation to fatigue by means of any rectangular prism circumscribing the stress path which, under a wide range of loading histories, has geometrical properties equivalent to the circumscribing ellipsoid:

Proposition: "Given an ellipsoid E in R^m with centre located at the origin and an arbitrary orthonormal basis $\{\mathbf{n}_i, i = 1,\ldots,m\}$ of R^m, let P be a rectangular prism circumscribing E such that its faces are orthogonal to each one of the basis elements. If $\lambda_i, i = 1,\ldots,m$ are the magnitudes of the principal semi-axes of E and $a_i, i = 1,\ldots,m$ denote the distances of the centre of the ellipsoid to the faces of the rectangular prism, then:

$$\sum_{i=1}^{5}\lambda_i^2 = \sum_{i=1}^{5}a_i^2 \ ." \tag{4}$$

Proof: (see Mamiya & Araújo (2000)).

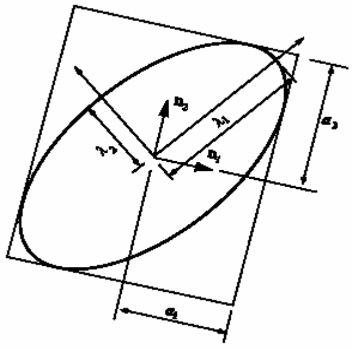


Figure 4. Invariance property of the rectangular prism circumscribing an ellipsoid.

The aforementioned statement is of fundamental importance for the computation of $f(\tau)$ since it precludes the need to determine the principal semi-axes of the ellipsoid. More specifically, the shear solicitation to fatigue would be measured as:

$$f(\tau) = \sqrt{\sum_{i=1}^{5} a_i^2} \; . \tag{5}$$

As a consequence, the calculation of $f(\tau)$ becomes a trivial task:

- For each time instant *t*, compute the Cauchy stress tensor $\sigma(t)$ and the corresponding deviatoric stress states $\mathbf{S}(t) = \sigma(t) - \frac{1}{3} tr\,\sigma(t)\,\mathbf{I}$;
- For each time instant *t*, determine its components in terms of any arbitrary orthonormal basis $\mathbf{N}_i$, $i = 1,\ldots,5$: $S_i(t) = (\mathbf{S}(t), \mathbf{N}_i)\; i = 1,\ldots,5$;
- Compute the amplitudes a_i, $i = 1,\ldots,5$ of the deviatoric stress as:

$$a_i = \frac{1}{2}\left(\max_t S_i(t) - \min_t S_i(t)\right)$$

- Compute the shear solicitation to fatigue as (5):

$$f(\tau) = \sqrt{\sum_{i=1}^{5} a_i^2}$$

The resulting fatigue criterion can be written as:

$$\sqrt{\sum_{i=1}^{5} a_i^2} + \kappa\, p_{\max} \le \lambda \, , \tag{6}$$

where $p_{\max}$ is the maximum value of the hydrostatic stress along the stress history while κ and λ are material parameters. The motivation to consider $g(\sigma) = p_{\max}$ as the measure of solicitation by normal stresses is based on the fact that the hydrostatic stress can be shown to be the average of the normal stress σ over all the planes passing through the material point.

3. Assessment of the fatigue models

In order to assess the new fatigue model and compare its performance with the classical Crossland criterion, a number of experimental data for three different steel alloys were collected from the literature. These data correspond to synchronous in phase and out of phase (shift angle β) biaxial fatigue tests involving normal and shear stresses under a critical state of solicitation. This essentially means that the amplitude and mean values of the normal and shear stresses (σ_a, σ_m and τ_a, τ_m) produced in each test were the maximum values that the specimen could withstand without breaking by fatigue up to a limit of $2x10^6$ cycles. Table 1 reports the data obtained by Nishihara and Kawamoto (1945) for a mild steel, while Tables 2 and 3 contain the data produced by Heidenreich et al. (1983) and Froustey & Lassere (1989) in 34Cr4 and 30NCD16, respectively. To implement the aforementioned multiaxial criteria, information concerning the fatigue limits under fully reversed bending, f_{-1}, and torsion, t_{-1}, are required. These data are also reported in Tables 1 to 3.

To quantitatively evaluate the quality of the results provided by our model, we define an *error index* as:

$$I := \frac{f(\tau)+g(\sigma)-\lambda}{\lambda}\times 100\ (\%)\,. \tag{7}$$

Table 1: Experimental and predicted results for hard steel (Nishihara & Kawamoto, 1945) $f_{\_1}$= 313.9MPa, $t_{\_1}$= 196.2MPa

Test N°	σ_a (MPa)	τ_a (MPa)	σ_m (MPa)	τ_m (MPa)	β(°)	I(%) Crossland	I(%) Our model
1	138.1	167.1	0	0	0	-2.26	-2.28
2	140.1	169.9	0	0	30	-2.54	-0.81
3	145.7	176.3	0	0	60	-3.59	2.93
4	150.2	181.7	0	0	90	-3.73	6.27
5	245.3	122.6	0	0	0	3.69	1.44
6	249.7	124.8	0	0	30	0.02	3.17
7	252.4	126.2	0	0	60	-8.34	4.3
8	258.0	129.0	0	0	90	-17.81	6.7
9	299.1	62.8	0	0	0	0.94	0.92
10	304.5	63.9	0	0	90	-2.98	2.74

Table 2: Experimental and predicted results for 34Cr4 (Heidenreich et al., 1983) $f_{\_1}$= 410MPa, $t_{\_1}$= 256MPa

Test N°	σ_a (MPa)	τ_a (MPa)	σ_m (MPa)	τ_m (MPa)	β(°)	I(%) Crossland	I(%) Our model
1	314	157	0	0	0	-0.51	-0.55
2	315	158	0	0	60	-12.3	-0.19
3	316	158	0	0	90	-22.7	0.08
4	315	158	0	0	120	-5.1	-0.19
5	224	224	0	0	90	-8.38	5.15
6	380	95	0	0	90	-7.32	0.37
7	316	158	0	158	0	0.54	0.08
8	314	157	0	157	60	-12.3	-0.64
9	315	158	0	158	90	-21.78	-0.11
10	279	140	279	0	0	-6.38	-6.38
11	284	142	284	0	90	-25.52	-4.83
12	212	212	212	0	90	-9.4	3.41

Table 3: Experimental and predicted results for 30NCD16 (Froustey & Lassere, 1989) $f_{\_1}$= 660MPa, $t_{\_1}$= 410MPa

Test N°	σ_a (MPa)	τ_a (MPa)	σ_m (MPa)	τ_m (MPa)	β(°)	I(%) Crossland	I(%) Our model
1	485.0	280.0	0	0	0	1.77	1.77
2	480.0	277.0	0	0	30	-27.27	0.7
3	480.0	277.0	300.0	0	60	3.91	3.91
4	480.0	277.0	300.0	0	90	-3.36	3.91
5	470.0	270.0	300.0	0	0	-11.33	1.50
6	473.0	273.0	300.0	0	30	-25.12	2.45
7	590.0	148.0	300.0	0	60	0.10	0.11
8	565.0	141.0	300.0	0	90	-7.55	-4.07
9	540.0	135.0	300.0	0	0	-14.97	-8.15
10	211.0	365.0	300.0	0	90	-0.68	-0.68

This index provides a measure of how close the prediction of the criterion is with respect to the experimental data. Once we are working with critical solicitation data, it seems clear that a negative I yields a non-conservative fatigue strength prediction, as the criterion assumes that further solicitation can be sustained by the component while data show the opposite.

Application of the proposed model to all tests reported in Tables 1 to 3 provided an error index that varied in the worst scenarios between -8.15 (test 9, Table 3) and 6.7 (test 8, Table 1), while the Crossland model predictions lied between -27.27 (test 2, Table 3) and 3.91 (test 3, Table3). These results emphasize the improvement obtained to estimate the fatigue strength when the current model is considered.

4. Computational aspects

An usual procedure considered in fatigue analysis is to recover, experimentally or numerically, the stress history at points of the mechanical component or of the structure, which are assumed to be critically loaded, and then to apply a fatigue criterion. Such points are often chosen as those associated with the highest levels of the Mises stresses, for instance. When multi-axial, non-proportional stress histories are observed, however, material points which are the most threatened by fatigue are not necessarily those associated with the highest levels of Mises stresses (or any other measure of equivalent stress), since the peaks of each stress components may eventually occur at very distinct time instants. As a consequence, an alternative to *a-priori* selection of critical points should be considered. This section describes the implementation of fatigue endurance criteria to an academic finite element code, which enables the analysis of the solicitation to fatigue over all the points of the structure in one single step, avoiding hence the subjective choice of critical points.

In our study, we implemented the Crossland criterion and the one described in this paper into the finite element code **ef++** (version 2.3), developed by the Group of Mechanics of Materials at the University of Brasília. The main features of this numerical code include:

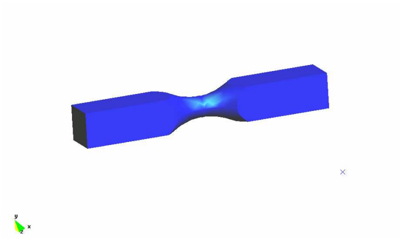
a) Elastic and inelastic behaviors under hypothesis of linear kinematics can be considered;
b) It is portable in the sense that it can be executed in both Linux, Windows or any other operating systems supporting GNU gcc compiler. It is developed under the copyright terms of the GNU General Public License as published by the Free Software Foundation;
c) The graphical interface is not included in the program, so as to allow an easier portability across platforms. Currently we are using the GiD pre and post-processor (gid.cimne.upc.es), developed at the International Centre of Numerical Methods in Engineering of the Universidad Politecnica da Cataluña (CIMNE-UPC).

In order to implement the fatigue criteria into **ef++**, the following features were included:

a) Proportional and non-proportional, piecewise affine or sinusoidal loading paths can be imposed;
b) The stress history is recorded for each nodal point of the mesh and for each time instant considered along the numerical simulation. Then, the fatigue criteria are applied to each nodal point of the mesh. As a consequence, the numerical procedures associated with the fatigue analysis does not have to be developed at element level;

5. Numerical results

In order to illustrate the use of the resulting finite element code for a fatigue endurance analysis, we considered the example of a specimen subjected to bending-torsion loading programs. We considered the specimen illustrated in Fig. 5, as described by Gough et al. (1951).



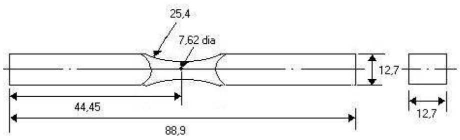


Figure 5. Specimen considered in the numerical simulation.

Discretization of the specimen, together with the grips attached at its ends, was performed with linear tetrahedral elements. During mesh generation, the mean size of the element edges was fixed as 1 mm close to the specimen neck and 9 mm elsewhere. The resulting mesh, illustrated in Fig. 6, defines 7,621 elements and 4,412 nodal points.

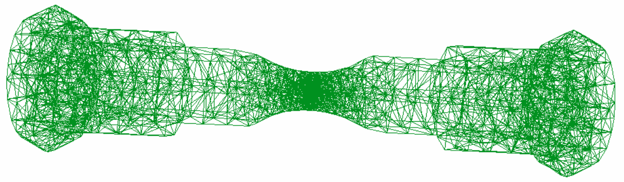


Fig. 6. Discretization of the specimen, defining 7,621 linear tetrahedral elements and 4,412 nodal points.

The left grip was rigidly fixed, while a force and a couple were considered at the right grip so as to impose the bending and the torsion solicitation upon the specimen. Three of the experimental data described in section 3 were considered in the numerical simulations (all stress values are expressed in MPa):

Case 1: Hard steel (Nishihara & Kawamoto, 1945), $f_{-1} = 313.9$, $t_{-1} = 196.2$:

$$\sigma(t) = 140.4 \sin \omega t\,, \qquad \tau(t) = 169.9 \sin(\omega\, t - 30^{\circ}) \tag{8}$$

Case 2: 34Cr4 (Heidenreich et al., 1983), $f_{-1} = 410$, $t_{-1} = 256$

$$\sigma(t) = 314 \sin \omega t\,, \qquad \tau(t) = 157 + 157 \sin(\omega\, t - 60^{\circ}) \tag{9}$$

Case 3: 30 NCD16 (Froustey & Lasserre, 1989), $f_{-1} = 660$, $t_{-1} = 410$:

$$\sigma(t) = 480 \sin \omega t\,, \qquad \tau(t) = 300 + 277 \sin(\omega\, t - 45^{\circ}) \tag{10}$$

Solicitation to fatigue was quantified at each nodal point by considering the *fatigue endurance index*, defined as:

$$I_{endur} := \frac{f(\tau) + g(\sigma) - \lambda}{\lambda} \times 100\,(\%)\,. \tag{11}$$

Although the right hand sides of expressions (11) and (7) are equal, they provide different information: the index error I measures the quality of the criteria when their results are compared to experimental data, while the endurance index I_{endur} informs the designer how close the stress history is to a situation of fatigue failure. If $I_{endur} \le 0$, it means that the stress levels are below the fatigue limit and endurance (number of cycles above 10^6) is expected. On the other hand, $I_{endur} > 0$ is associated with fatigue failure after a finite number of cycles.

Remark: Fatigue endurance within the setting of the criteria considered in this paper is not dependent on the concept of cycle, which is very difficult to define when stress histories other than harmonic ones are considered.

Figure 7 shows the distribution of the σ_{xx} normal stress component acting on the specimen of the case 1 at time instant $t = \dfrac{\pi}{2\,\omega}$, i.e., at one fourth of the period. After o complete cycle (partitioned in 32 time intervals), the fatigue endurance index I_{endur} can be computed, resulting in the mapping illustrated in Fig. 8 when the Crossland criterion is considered. In this case, the critical value of the index I_{endur} computed by the program is $-2.52\,\%$, while the calculated analytical value is $-2.63\,\%$. Figure 9 depicts the same field, but only the regions where the endurance index attains values above $-10\,\%$ are mapped.

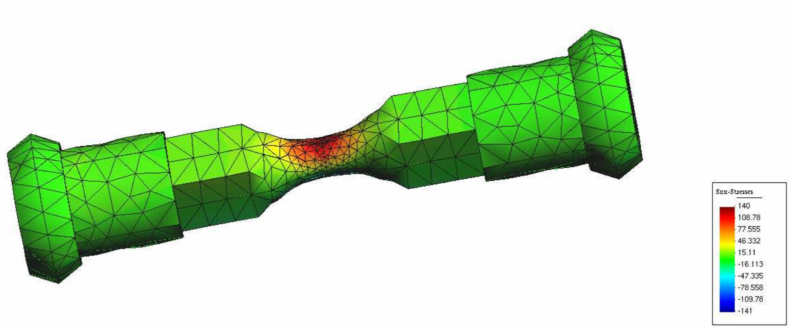


Figure 7. Distribution of the σ_{xx} stress component acting on the specimen of the **case 1** at time instant $t = \dfrac{\pi}{2\omega}$.

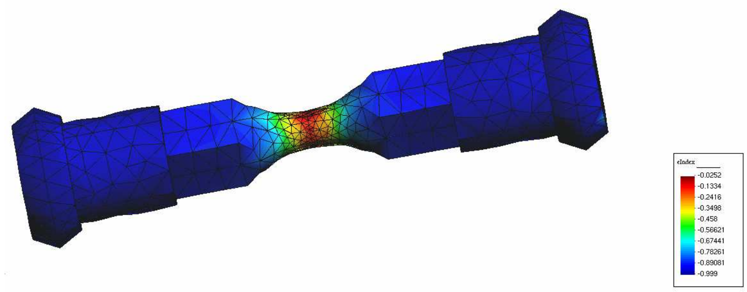


Figure 8. Distribution of the fatigue endurance index I_{endur} associated with the Crossland criterion, for **case 1**. The critical value computed by the program is $I_{endur} = -2.52\,\%$.

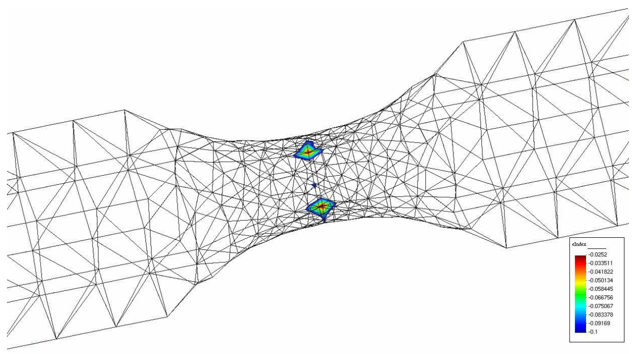


Figure 9. Distribution of the fatigue endurance index I_{endur} associated with the Crossland criterion, for **case 1**. Only values of the index above $-10\,\%$ are mapped.

Figures 10 shows the endurance index field for the same case, when the criterion presented in this paper is considered. The peak value of the index computed by the program was $-1.08\,\%$, while the expected value was $-0.81\,\%$.

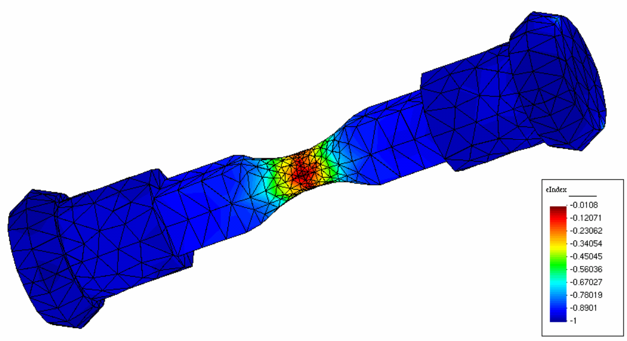


Figure 10. Distribution of the fatigue endurance index *I* associated with the Crossland criterion, for **case 1**. The critical value computed by the program is $I_{endur} = -2.52\,\%$.

The results presented above, together with the ones corresponding to cases 2 and 3, are summarized in Table 4. The differences between the peak values of the endurance index computed by the finite element program and those obtained analytically is due to the approximations involved in the calculation of the stresses when the numerical method is considered.

Table 4. Peak fatigue endurance index I_{endur} for the three cases of bending-torsion tests, provided by the finite element program and by considering analytically obtained stress paths.

Case	*I (%) Crossland (MEF)*	*I (%) Crossland (Analytical)*	*I (%) Mamiya & Araújo (MEF)*	*I (%) Mamiya & Araújo (Analytical)*
1	-2.52	-2.63	-1.08	-0.81
2	-13.6	-12.3	-1.49	-0.64
3	-3.69	-7.8	-3.28	3.91

6. Final remarks

A new multiaxial criterion to evaluate the strength of hard metals under a high cycle fatigue regime has been proposed. Assessment of the criterion against a number of experimental data for different materials showed a very good agreement. Further, the predictions obtained by our model were significantly better than those provided by Crossland, and in particular under nonproportional loading conditions. A very interesting feature that should be stressed about our model is that it is extremely simple to implement. The criteria considered in this paper were implemented in the academic **ef++** finite element code, providing a powerful tool to design against fatigue.

7. Acknowledgments

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8. References

Akrache, R., & Lu, J., 1999, Three-dimensional calculations of high cycle fatigue life under out-of-phase multiaxial loading, Fatigue Fract Engng Mater Struct, Vol. 22, pp.527-534.

Bin Li, Santos, J. L. & Freitas, M., 2000, A unified numerical approach for multiaxial fatigue limit evaluation, Mech. Struct. & Mach., Vo. 28, pp.85-103.

Crossland, B., 1956, "Effect of large hydrostatic pressures on the torsional fatigue strength of an alloy steel", Proc. Int. Conf. on Fatigue of Metals, IMechE, London, pp. 138-149.

Deperrois, A., 1991, "Sur le calcul de limites d'endurance des aciers", Thèse de Doctorat, Ecole Polytechnique, Paris.

Froustey, C. & Lassere, S., 1989, Multiaxial Fatigue Endurance of 30NCD16 Steel, Int. J. Fatigue, Vol. 11, pp. 165-178.

Goodman, J., 1899, Mechanics applied to engineering, Longmans Green, London.

Gough, H. J., Pollard, H. V. & Cleushaw, W. J., 1951, Some experiments on the resistance of metals to fatigue under combined stress, Memo 2522, Aeronautical Research Council, HMSO, London.

Hendenreich, R., Zenner, H. & Hichter, I., 1983, Dauerschwingfestigkeit bei mehrachsiger Beanspruchung. Forschungshefte FKM, Heft 105.

Mamiya, E. N. & Araújo, J. A., 2002, Fatigue limit under multiaxial loadings: on the definition of the equivalent shear stress. Mechanics Research Communications, Vol.29, pp.141-151.

Milne, I., 1994, The Importance of the Management of Structural Integrity, Engineering Failure Analysis, Vol. 1, No 3, pp. 171-181.

Nishihara, T. & Kawamoto, M., 1941, The strength of metals under combined alternating bending and torsion, Memoirs of the College of Engineering, Kyoto Imperial University, Vol. 10, pp.177–201.

Papadopoulos, I. V., Davoli, P., Gorla, C., Filippini, M. & Bernasconi, A., 1997, A comparative study of multiaxial high-cycle fatigue criteria for metals, Int. J. Fatigue, Vol.19, pp.219–235.

Reed, R. P., Smith, J. H. & Christ, W., 1983, The Economic Effects of Fracture in the United States: Part 1, Spec Pub. No 647-1, U.S. Department of Commerce, National Bureau of Standards, U.S. Government Printing Office, Washington DC.

Schütz, W., 1996, A history of fatigue, Engng. Farcture Mech., Vol. 54, pp. 263-300.

Sines, G. & Ohgi, G., 1981, Fatigue criteria under combined stresses or strains, Trans. ASME J. of Engng. Mater. Technolgy, Vol. 103, pp. 82-90.

Socie, D., 1993, Critical plane approaches for multiaxial fatigue damage assessment. In: Advances in Multiaxial Fatigue (Eds.: McDowell, D. L. & Ellis, R)., Special Technical Publication 1191, pp.7-36, Philadelphia, PA. ASME.

Wöhler, A.,1860, Versuche über die Festigkeit der Eisenbahnwagenachsen, Zeitschrift für Bauwesen, 10; English summary (1867), Engineering Vol.4, pp.160-161.