FORCED SINGULAR PERTUBATIONS IN MULTIRATE TERRESTRIAL STRAPDOWN NAVIGATION

Jacques Waldmann
Instituto Tecnológico de Aeronáutica - Department of Systems and Control - 12228-900 - São José dos Campos - Brasil
e-mail: jacques@ele.ita.br

Abstract. *A forced singular perturbations (FSP) approach is proposed as a theoretical frame to Bar-Itzhack's multiple time-scale solution to the problem of terrestrial strapdown navigation. The approach splits the navigation equations into body and local-level reference frames to reduce the computational workload. The velocity vector is decomposed into two components, each in its own particular reference frame: the inertial thrust velocity in the body frame, driven by nongravitational input, and the ground velocity in the local-level frame, affected by the influence of gravity. The dynamics is then forced into two distinct boundary layers and an outer solution. Thrust velocity dynamics is solved at a fast rate, gravity-influenced ground velocity at an intermediate rate, and position at a slow rate. Limitations of FSP as a theoretical justification of the approach are indicated in terms of violation of the matching conditions across the boundary layers. Forcing the dynamics into distinct time scales caused a degradation of the continuous time solution. The loss of accuracy was not significant in comparison with the covered distance. Discretization and use of inertial data in incremental form, however, had a significant impact on navigation acuracy. The evaluation of performance in a variety of conditions indicated that, for given fast inertial data acquisition and slow position update rates subject to constrained computational throughput, the multirate algorithm showed superior accuracy in comparison with the simultaneous integration of velocity and position dynamics at the slow rate. The results are valuable in the design of autonomous vehicles expected to accomplish navigation by purely inertial means for brief periods of time.*

Keywords*. Strapdown inertial navigation, forced singular perturbations, navigation error, autonomous vehicles, robotics.*

1. Introduction

Forced singular perturbations (FSP) are often used in real-time suboptimal control of nonlinear dynamic systems to avoid solving a two-point boundary-value problem (Kokotovic *et allii*, 1976; Shinar, 1983; Sridhar and Gupta, 1980; Tihonov, 1952). The latter imposes a high computational burden in problems with a large state dimension, yields an open-loop control law, and does not render a real-time closed-loop synthesis. FSP reduces complexity by artificially separating the fast state components into distinct time scales (Ardema and Rajan, 1985a, 1985b; Kelley, 1971; Kokotovic *et allii*, 1980). It is accomplished by multiplication of the time derivatives by successive powers of a small positive perturbation parameter ε. Smaller interconnected static optimization problems arise and a distinct solution to each one of the fast variables is then sought within the corresponding time scale (Calise, 1978; Kelley, 1971, 1973). Full-order system complexity is reduced; nevertheless, additional algebraic constraints arise. In general, the solution in a fast time scale does not match the corresponding initial condition of the fast-varying variable. To cope with such discontinuities, asymptotic expansion methods are employed to match the solutions across the distinct time scales, as the composed boundary layer and stream solution in fluid mechanics (Ardema, 1976; Kelley and Edelbaum, 1970; Wasow, 1965; Freedman and Granoff, 1976). The composed solution to the singularly perturbed system is expected to be uniformly valid across the boundary layers. For that purpose, the solution of the reduced-order problem is corrected across consecutive boundary layers, and suboptimal closed-loop control is attainable (Shinar, 1983; Calise, 1981; Shinar, 1985). The approach calls for an appropriate ordering of the state components according to how fast they behave relative to one another. This ordering is frequently based on the observation of the physical problem.

The objective in the present application of FSP *is not* the synthesis of a suboptimal closed-loop control law. Rather, one seeks the order reduction that occurs within each boundary layer to reduce the computational workload of real–time integration of the navigation equations, while concurrently avoiding significant errors. Recent advances in autonomous robotic vehicles motivate this research. The vehicle is assumed to be equipped with strapdown inertial sensors that provide the capability of determining its position and velocity for a period of time. It is expected to accomplish its goals autonomously with limited computational resources, and resort to external aids only when required to limit the navigation errors that arise from processing purely inertial data. Among the advantages of a strapdown inertial navigation system (SDINS), one can point out that it is self-contained, obviates irradiation to and/or collection of energy from the environment, is suitable for autonomous navigation of robotic vehicles, and is naturally hardened to external interference.

By looking into SDINS dynamics, the vehicle maneuverability indicates that the motion of the body coordinate frame S_b should be accounted for in the fastest boundary layer. Contrastingly, the local-level reference frame S_l used to describe terrestrial velocity estimates - East, North and Down components of ground velocity, rotates at a much smaller rate. This observation suggests that S_l motion should be considered in an intermediate boundary layer. Finally, ground position coordinates - latitude, longitude and altitude, evolve at an even slower rate. Hence, it is reasonable to group ground position coordinates in the reduced-order problem, also known as the outer solution.

In a well-posed problem, the full-order solution with ε=1 is in the neighborhood of the zeroth-order outer solution obtained by setting ε=0 (Tihonov, 1952). Besides reducing the computational burden, the zeroth-order solution does not require knowledge of the value of ε used to force the fast state variables into distinct boundary layers. In practice, the reduced-order solution is valid when it is consistent with the physics of the full order problem.

Next are derived the velocity equations in body and local level frames, S_b and S_l, respectively. Assumptions are made to force the problem dynamics into distinct time scales and to assure the linearity of the velocity equations, so that the superposition principle applies. Eigenvalue analysis is used to qualitatively justify the casting of the strapdown terrestrial navigation equations into an FSP formulation. Because suboptimal closed-loop control is not pursued here, the boundary layers contain more than one fast scalar variable. Limitations of the FSP approach as a theoretical background to a multirate algorithm are pointed out in terms of violations of the asymptotic matching conditions across the boundary layers. An *ad-hoc* modification based on an Euler approximation to the continuous-time multirate scheme proposed by Bar-Itzhack (1977, 1978) results in a discrete-time algorithm with vertical channel damping. The dynamics of the thrust velocity represented in the body frame S_b is integrated numerically using the fastest computational rate. The ground velocity represented in the local-level frame S_l is integrated using the intermediate rate, and likewise in the slowest rate the reduced outer problem describing geographic position dynamics represented in the earth-fixed frame S_e. A numerical investigation of the impact of physical separation into distinct time scales on navigation error is carried out, and the results compared with those obtained from the simultaneous integration of the navigation equations at the slowest rate.

2. The navigation equations in the body frame

Consider a vehicle that moves relative to frame S_e fixed to the rotating earth. Following the notation in Bar-Itzhack (1977 and 1978), one is interested in having estimates of position vector $\mathbf{R}$ relative to the center of the earth, and of ground velocity vector $\mathbf{U} = \overset{e}{\mathbf{R}}$. The superscript indicates the reference frame in which the time derivative vector is observed. The accelerometers are fixed to the vehicle's body frame S_b and measure the specific force $\mathbf{f}$ acting on the vehicle relative to the inertial frame S_i:

$$\mathbf{f} = \overset{ii}{\mathbf{R}} - \mathbf{g_m} \tag{1}$$

where $\mathbf{g_m}$ represents the local gravitation vector. Accelerometer output is often given as a velocity increment between samples represented in the body frame, i.e., $\Delta\boldsymbol{\beta}_\mathbf{b} = \int_{(k-1)T_{gyr}}^{kT_{gyr}} \mathbf{f_b}(\tau)d\tau$. Subscripts indicate the reference frame used to represent a vector quantity. Rate gyros measure the angular rate vector $\boldsymbol{\omega}^{\mathbf{bi}}$ represented in the body frame, and their output is frequently an incremental angular displacement, i.e., $\Delta\boldsymbol{\phi}_\mathbf{b} = \int_{(k-1)T_{gyr}}^{kT_{gyr}} \boldsymbol{\omega}_\mathbf{b}^{\mathbf{bi}}(\tau)d\tau$. Inertial velocity $\overset{i}{\mathbf{R}} = \mathbf{V}$ relates to the time derivative of position $\overset{b}{\mathbf{R}}$ seen by an observer in the body frame according to Bar-Itzhack:

$$\overset{i}{\mathbf{V}} = \overset{b}{\mathbf{V}} + \boldsymbol{\omega}^{\mathbf{bi}} \times \mathbf{V} \tag{2}$$

Rearranging Eqs. (1) and (2) yields:

$$\overset{b}{\mathbf{V}} = -\boldsymbol{\omega}^{\mathbf{bi}} \times \mathbf{V} + \mathbf{f} + \mathbf{g_m} \tag{3}$$

Damping of the unstable vertical channel is accomplished by adding an altitude-dependent term $\mathbf{s}$(h) derived from altimeter data to that channel's dynamics (Siouris, 1993). To describe the vector quantities in Eq.(3) in the body frame, one can write:

$$\overset{b}{\mathbf{V}}_\mathbf{b} = \dot{\mathbf{V}}_\mathbf{b} = -[\omega^{\mathbf{bi}}]_\mathbf{b}\mathbf{V_b} + \mathbf{f_b} + \mathbf{g_{m,b}} + \mathbf{s}(h) \tag{4}$$

$$[\boldsymbol{\omega}^{\mathbf{bi}}]_\mathbf{b} = \begin{bmatrix} 0 & -\boldsymbol{\omega}_{zb}^{bi} & \boldsymbol{\omega}_{yb}^{bi} \\ \boldsymbol{\omega}_{zb}^{bi} & 0 & -\boldsymbol{\omega}_{xb}^{bi} \\ -\boldsymbol{\omega}_{yb}^{bi} & \boldsymbol{\omega}_{xb}^{bi} & 0 \end{bmatrix}; \qquad \mathbf{V_b} = \begin{bmatrix} V_{xb} \\ V_{yb} \\ V_{zb} \end{bmatrix}$$

The gravitation term represented in the body frame relates to the local-level gravity model according to:

$$\mathbf{g_{m,b}} = \mathbf{D_b^l}(\mathbf{g_l} + [\mathbf{\Omega}]_l^2 \mathbf{D_l^e R_e}); \quad \mathbf{\Omega_l} = \boldsymbol{\omega}_l^{ei} = \begin{bmatrix} \Omega\cos(\lambda) \\ 0 \\ -\Omega\sin(\lambda) \end{bmatrix}; \quad \mathbf{g_l} = [0 \;\; 0 \;\; g(\lambda,h)]^T \tag{5}$$

where $g(\lambda,h)$ is given by the U.S. Department of Defense World Geoid System WGS-84 model (Siouris, 1993) and $\mathbf{D_b^l}$ is the direction cosine matrix (DCM) from S_l to S_b. The initial condition to solve Eq.(4) is determined by:

$$\mathbf{V_b}(0) = \mathbf{D_b^l}(0)[\mathbf{U_l}(0) + \mathbf{D_l^e}(0)[\mathbf{\Omega}]_e \mathbf{R_e}(0)]; \quad \mathbf{U_l}(0) \text{ given} \tag{6}$$

where e denotes the earth-fixed coordinate frame, and $\mathbf{U_l}(0)$ is the known initial ground velocity represented in the local-level frame $[V_N(0) \; V_E(0) \; V_D(0)]^T$. The initial direction cosine matrices are assumed available from an initial alignment procedure. Regarding earth-fixed position, the following relation holds:

$$\overset{i}{\dot{\mathbf{R}}} = \mathbf{V} = \overset{e}{\dot{\mathbf{R}}} + \mathbf{\Omega}\times\mathbf{R} = \mathbf{U} + \mathbf{\Omega}\times\mathbf{R} \tag{7}$$

After expressing the last equation in the earth-fixed frame, it is possible to compute the earth-fixed position as follows:

$$\overset{e}{\dot{\mathbf{R}}}_e = \dot{\mathbf{R}}_e = \mathbf{U_e} = \mathbf{D_e^b V_b} - [\Omega]_e \mathbf{R_e}; \quad \mathbf{R_e}(0) \text{ given} \tag{8}$$

where $\mathbf{R_e}(0)$ is the known initial ground position. The above differential equation of the ground position is solved simultaneously with Eq. (4). Local-level and earth-fixed representations of ground velocity are related as follows:

$$\mathbf{U_l} = [V_N \;\; V_E \;\; V_D]^T = \mathbf{D_l^e U_e} = \mathbf{D_l^e}(\mathbf{D_e^b V_b} - [\mathbf{\Omega}]_e \mathbf{R_e})$$

where $\mathbf{D_l^e}$ is determined from the estimated latitude and longitude. As an alternative to the integration of Eq. (8), latitude, longitude and altitude coordinates in the earth-fixed reference frame can be computed from:

$$\begin{aligned} &\dot{\lambda} = V_N / [R_N(\lambda) + h] \\ &\dot{\Lambda} = V_E / \{[R_E(\lambda) + h]\cos(\lambda)\} \\ &\dot{h} = -V_D + s'(h) \\ &\lambda(0); \Lambda(0); h(0) \text{ given} \end{aligned} \tag{9}$$

where $s'(h)$ is a vertical-channel altitude-dependent damping term (Siouris, 1993), and R_N and R_E are the earth's curvature radii in the north-south and east-west directions, respectively (see Table I). One is interested in determining real-time estimates of V_N, V_E, V_D, λ, Λ, and h by integration of Eqs. (4) and (9) with sampled incremental data provided by strapdown rate gyros and accelerometers. The direction cosine matrix $\mathbf{D_b^l}$ in Eq. (5) is required at a high update rate, which imposes a significant computational workload.

3. The navigation equations in the local-level frame

The inertial acceleration can be obtained from Eq. (7) as follows:

$$\overset{i}{\dot{\mathbf{V}}} = \overset{i}{\dot{\mathbf{U}}} + \overset{i}{\dot{\mathbf{\Omega}}}\times\mathbf{R} + \mathbf{\Omega}\times\mathbf{V} \tag{10}$$

where the inertial time derivative of the ground velocity $\overset{i}{\dot{\mathbf{U}}}$ is related to its local-level rate $\overset{l}{\dot{\mathbf{U}}}$ according to:

$$\overset{i}{\dot{\mathbf{U}}} = \overset{l}{\dot{\mathbf{U}}} + (\mathbf{\Omega} + \boldsymbol{\rho})\times\mathbf{U}; \quad \boldsymbol{\rho} = \boldsymbol{\omega}^{le} \tag{11}$$

Substitution of Eqs. (7) and (11) in Eq. (10) yields:

$$\overset{i}{\dot{\mathbf{V}}} = \overset{l}{\dot{\mathbf{U}}} + (2\mathbf{\Omega} + \boldsymbol{\rho})\times\mathbf{U} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R}) \tag{12}$$

From Eq. (1) and recalling that $\overset{\mathbf{i}}{\dot{\mathbf{R}}} = \mathbf{V}$, the ground velocity time rate as observed in the local-level frame is then:

$$\overset{\mathbf{l}}{\dot{\mathbf{U}}} = -(2\boldsymbol{\Omega} + \boldsymbol{\rho}) \times \mathbf{U} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) + \mathbf{g_m} + \mathbf{f} \tag{13}$$

Representing the above vector equation in the local-level frame yields:

$$\overset{\mathbf{l}}{\dot{\mathbf{U}}}_{\mathbf{l}} = \dot{\mathbf{U}}_{\mathbf{l}} = -[2\Omega + \rho]_{\mathbf{l}} \mathbf{U}_{\mathbf{l}} - [\Omega]_{\mathbf{l}}^2 \mathbf{R}_{\mathbf{l}} + \mathbf{g}_{\mathbf{m},\mathbf{l}} + \mathbf{f}_{\mathbf{l}} + \mathbf{s}''(\mathrm{h}); \quad \mathbf{U}_{\mathbf{l}}(0) \text{ given} \tag{14}$$

$$[2\boldsymbol{\Omega} + \boldsymbol{\rho}]_{\mathbf{l}} = \begin{bmatrix} 0 & (2\Omega + \dot{\Lambda})\sin(\lambda) & -\dot{\lambda} \\ -(2\Omega + \dot{\Lambda})\sin(\lambda) & 0 & -(2\Omega + \dot{\Lambda})\cos(\lambda) \\ \dot{\lambda} & (2\Omega + \dot{\Lambda})\cos(\lambda) & 0 \end{bmatrix}; \quad \mathbf{g}_{\mathbf{l}} = -[\boldsymbol{\Omega}]_{\mathbf{l}}^2 \mathbf{R}_{\mathbf{l}} + \mathbf{g}_{\mathbf{m},\mathbf{l}} = \begin{bmatrix} 0 & 0 & \mathrm{g}(\lambda, \mathrm{h}) \end{bmatrix}^{\mathrm{T}}$$

where $\mathbf{s}''(\mathrm{h})$ is a vertical-channel altitude-dependent damping term. Time integration of Eq. (14) yields estimates of V_N, V_E, and V_D. Its solution demands the transformation of specific force measurements from the body frame to the local-level frame:

$$\mathbf{f}_{\mathbf{l}} = \mathbf{D}_{\mathbf{l}}^{\mathbf{b}} \mathbf{f}_{\mathbf{b}} \tag{15}$$

The earth-fixed position estimate can then be obtained from the solution to Eq. (14):

$$\mathbf{U}_{\mathbf{e}} = \dot{\mathbf{R}}_{\mathbf{e}} = \mathbf{D}_{\mathbf{e}}^{\mathbf{l}} \mathbf{U}_{\mathbf{l}}; \quad \mathbf{R}_{\mathbf{e}}(0) \text{ given} \tag{16}$$

Alternatively, substitution of $\mathbf{U}_{\mathbf{l}}$ obtained from Eq. (14) in Eq. (9) produces the earth-fixed position estimates λ, Λ, and h. Again, the computation of $\mathbf{D}_{\mathbf{b}}^{\mathbf{l}}$ in Eq. (15) is required prior to updating the solution of Eq. (14).

4. Forced Singular Perturbations

Equations (4) and (14) describe the evolution of inertial and ground velocities, respectively, in distinct coordinate frames − body and local-level frames, respectively. Likewise, Eqs. (8) and (16), or (9), characterize the earth-fixed position. The coordinate frames rotate at significantly different angular rates and our interest is to make use of this fact to reduce the computational workload of the numerical solution. The FSP approach is employed to provide a theoretical frame to a multirate integration scheme. Limitations of the approach are then pointed out along the derivation.

4.1. The Fast Boundary Layer: Thrust Velocity and Body Frame Equations

The description of the navigation equations in the body frame, which is assumed to rotate much faster than the local level one, is forced into the fastest time scale $t^b = t/\varepsilon^2$ where ε is a small positive parameter. The local level representation of the ground velocity is forced into an intermediate boundary layer with time scale $t^l = t/\varepsilon$. This is qualitatively justified by the eigenvalues of matrix $[\boldsymbol{\omega}^{\mathbf{bi}}]_{\mathbf{b}}$ in Eq. (4), namely $\{0, \pm j|\boldsymbol{\omega}^{\mathbf{bi}}|\}$, the complex pair having a magnitude much larger than those of $[2\boldsymbol{\Omega} + \boldsymbol{\rho}]_{\mathbf{l}}$ in Eq.(14), which are $\{0, \pm j|2\boldsymbol{\Omega} + \boldsymbol{\rho}|\}$, because the vehicle's maneuverability allows its body rotation relative to the inertial frame to be much faster than the rotation of its local-level frame. Actually, the eigenvalues of the coupled position and velocity linearized error equations are close to these ones (Bar-Itzhack and Berman, 1988). Hence, the assumptions made when forcing thrust velocity dynamics into the fastest boundary layer are physically meaningful. From the previous inspection of the eigenvalues, however, such forced separation diverges from reality when the vehicle is resting. A numerical investigation of such condition is later carried out. Now, recalling Eqs. (4), (8), and (14), the perturbed equations are:

$$\dot{\mathbf{R}}_{\mathbf{e}} = \frac{d\mathbf{R}_{\mathbf{e}}}{dt} = \mathbf{D}_{\mathbf{e}}^{\mathbf{b}} \mathbf{V}_{\mathbf{b}} - [\boldsymbol{\Omega}]_{\mathbf{e}} \mathbf{R}_{\mathbf{e}}$$

$$\varepsilon \dot{\mathbf{U}}_{\mathbf{l}} = \varepsilon \frac{d\mathbf{U}_{\mathbf{l}}}{dt} = -[2\Omega + \rho]_{\mathbf{l}} \mathbf{U}_{\mathbf{l}} - [\Omega]_{\mathbf{l}}^2 \mathbf{R}_{\mathbf{l}} + \mathbf{g}_{\mathbf{m},\mathbf{l}} + \mathbf{f}_{\mathbf{l}} + \mathbf{s}''(\mathrm{h});$$

$$\varepsilon^2 \dot{\mathbf{V}}_{\mathbf{b}} = \varepsilon^2 \frac{d\mathbf{V}_{\mathbf{b}}}{dt} = -[\boldsymbol{\omega}^{\mathbf{bi}}]_{\mathbf{b}} \mathbf{V}_{\mathbf{b}} + \mathbf{f}_{\mathbf{b}} + \mathbf{g}_{\mathbf{m},\mathbf{b}} + \mathrm{s}(\mathrm{h})$$

The zeroth-order solution in the fast boundary layer's time scale t^b is obtained by letting ε approach zero, and hence:

$$\frac{d\mathbf{R_e}}{dt^b}=\mathbf{0};\quad \mathbf{R_e}(t^b)=\mathbf{R_e}(0);\quad t^b\in[0,\infty) \tag{17a}$$

$$\frac{d\mathbf{U_l}}{dt^b}=\mathbf{0};\quad \mathbf{U_l}(t^b)=\mathbf{U_l}(0) \tag{17b}$$

$$\frac{d\mathbf{V_b}}{dt^b}=-[\boldsymbol{\omega}^{\mathbf{bi}}]_{\mathbf{b}}\mathbf{V_b}+\mathbf{f_b}+\mathbf{g}_{\mathbf{m,b}}(\mathbf{R}(0))+\mathbf{s}(\mathrm{h}(0)) \tag{18}$$

The above shows that both earth-fixed position and ground velocity remain frozen in this boundary layer, which considers the evolution of $\mathbf{V_b}$ dynamics only. Hence, Eq. (18) is linear in $\mathbf{V_b}$ and is driven by time-varying input $\mathbf{f_b}$, while $\mathbf{g_{m,b}}(\mathbf{R}(0))$ and $\mathbf{s}(h(0))$ become constant input signals. Consequently, Eq. (18) is now analyzed in terms of the superposition principle valid for linear dynamic systems. The inertial velocity $\mathbf{V_b}$ is then partitioned into a sum of two terms: the thrust velocity $\mathbf{V_{f,b}}$ that arises from all nongravitational forces acting on the vehicle with zero initial condition, and the ground velocity $\mathbf{V_{g,b}}$ caused by gravitation forces acting on the vehicle in addition to its initial inertial velocity. Hence:

$$\frac{d\mathbf{V_{f,b}}}{dt^b}=-[\boldsymbol{\omega}^{\mathbf{bi}}]_{\mathbf{b}}\mathbf{V_{f,b}}+\mathbf{f_b};\quad \mathbf{V_{f,b}}(0)=\mathbf{0} \tag{19}$$

$$\frac{d\mathbf{V_{g,b}}}{dt^b}=-[\boldsymbol{\omega}^{\mathbf{bi}}]_{\mathbf{b}}\mathbf{V_{g,b}}+\mathbf{g}_{\mathbf{m,b}}(\mathbf{R}(0))+\mathbf{s}(\mathrm{h}(0));\quad \mathbf{V_{g,b}}(0)=\mathbf{V_b}(0) \tag{20}$$

$$\mathbf{V_b}=\mathbf{V_{f,b}}+\mathbf{V_{g,b}} \tag{21}$$

where the initial conditions have been selected to eliminate the need to represent the ground velocity in the body coordinate frame (Bar-Itzhack, 1977, 1978). This boundary layer considers only the thrust velocity $\mathbf{V_{f,b}}$ dynamics described by Eq. (19). Its numerical integration should be performed at the fast computation rate T_{gyr} that captures the dynamics of the vehicle's maneuvers. Velocity and angular increments $\boldsymbol{\Delta\beta_b}$ and $\boldsymbol{\Delta\phi_b}$ from the inertial sensors will be employed in a straightforward manner in the discrete-time solution. In the fast boundary layer, $\mathbf{V_{f,b}}$ is expected to evolve from $\mathbf{V_{f,b}}(0)=\mathbf{0}$, and then converge to its initial value $\mathbf{V_{f,b}}^{\mathbf{l}}(0)$ at the onset of the intermediate boundary layer. The asymptotic matching condition from the fast to the intermediate boundary layers is written from Eq.(19) as:

$$\lim_{t^b\to\infty}\int_0^{t^b}\{-[\boldsymbol{\omega}^{\mathbf{bi}}]_{\mathbf{b}}\mathbf{V_{f,b}}+\mathbf{f_b}\}dt^b=\mathbf{V_{f,b}}^{\mathbf{l}}(0)$$

It will be seen later that the above condition is violated. In the following, the gravitational effects on the vehicle´s motion are considered in the intermediate boundary layer.

4.2. The Intermediate Boundary Layer: Gravitation Effects and Local-Level Frame Equations

The time scale is $t^l=t/\varepsilon$ in the intermediate boundary layer. The time-varying nature of the nongravitational force $\mathbf{f}$ has been taken into account in the fast boundary layer, and hence $\mathbf{f}=\mathbf{f}^{\mathbf{l}}(t^l)$ is considered to be constant in this time scale. The same rationale is used to assume a constant body angular rate $\boldsymbol{\omega}^{\mathbf{bi},\mathbf{l}}(t^l)$. On the other hand, the ground velocity $\mathbf{U_l}$ is assumed to fully develop within this time scale. The zeroth-order solution in this boundary layer's time scale t^l is produced by perturbing Eqs. (4), (8), (14), and by letting ε approach zero in the t^l time scale:

$$\frac{d\mathbf{R_e}}{dt^l}=\mathbf{0};\quad \mathbf{R_e}(t^l)=\mathbf{R_e}(0);\quad t^l\in[0,\infty) \tag{22a}$$

$$\mathbf{0}=-[\boldsymbol{\omega}^{\mathbf{bi}}(t^l)]_{\mathbf{b}}\mathbf{V_{f,b}}^{\mathbf{l}}(t^l)+\mathbf{f_b}^{\mathbf{l}}(t^l) \tag{22b}$$

$$\frac{d\mathbf{U_l}}{dt^l}=-[2\boldsymbol{\Omega}+\boldsymbol{\rho}]_{\mathbf{l}}\mathbf{U_l}-[\boldsymbol{\Omega}]_{\mathbf{l}}^2\mathbf{R_l}(0)+\mathbf{g}_{\mathbf{m,l}}(\mathbf{R}(0))+\mathbf{f_l}^{\mathbf{l}}(t^l)+\mathbf{s}''(\mathrm{h}(0)) \tag{23}$$

$\mathbf{V_{f,b}}^{\mathbf{l}}(t^l)$ represents the solution of the fast thrust velocity variable in the intermediate boundary layer, and should be obtained from the constraint in Eq.(22b). The ground velocity $\mathbf{U_l}$ is now decomposed into a sum of two terms: $\mathbf{U_{f,l}}$, arising from all nongravitational forces acting on the vehicle, and $\mathbf{U_{g,l}}$, caused by gravitation forces. Position $\mathbf{R_l}$ is analogously separated into components arising from gravitation and nongravitational forces, $\mathbf{R_{f,l}}$ and $\mathbf{R_{g,l}}$, respectively (see Bar-Itzhack, 1977, Eq.(32)). In order to linearize the ground velocity equation described by Eq. (23), it is assumed that $\boldsymbol{\rho}=\boldsymbol{\rho}(\mathbf{R},\mathbf{U})$ is constant within time scale t^l. This assumption is physically justified because $\mathbf{R}$ is frozen in this boundary layer, and $\mathbf{U}$ remains practically constant within a short computation interval. Hence, the angular rate $\boldsymbol{\rho}$ of the local-level frame relative to the rotating earth does not change significantly in this time scale. The linearity assumption allows the use of superposition in the intermediate boundary layer and simplifies the analysis of the effect of gravitation on ground velocity. Under this assumption, Eq. (23) is linear in $\mathbf{U_l}$, whereas $\mathbf{g_{m,l}}(\mathbf{R}(0))-[\mathbf{\Omega}]_l^2\mathbf{R_l}(0)=\mathbf{g_l}(\mathbf{R}(0))$, $\mathbf{f_l}^{\mathbf{l}}(t^l)$, and $\mathbf{s''}(h(0))$ are constant input signals in this time scale. Hence:

$$\frac{d\mathbf{U}_{f,l}}{dt^l} = -[2\mathbf{\Omega}+\boldsymbol{\rho}]_l\mathbf{U}_{f,l} - [\mathbf{\Omega}]_l^2\mathbf{R}_{f,l}(0) + \mathbf{f_l}^{\mathbf{l}}(t^l) \tag{24}$$

$$\frac{d\mathbf{U}_{g,l}}{dt^l} = -[2\mathbf{\Omega}+\boldsymbol{\rho}]_l\mathbf{U}_{g,l} - [\mathbf{\Omega}]_l^2\mathbf{R}_{g,l}(0) + \mathbf{g}_{m,l}(\mathbf{R}(0)) + \mathbf{s}''(h(0)) \tag{25}$$

$$\mathbf{U_l} = \mathbf{U}_{g,l} + \mathbf{U}_{f,l}; \qquad \mathbf{R_l} = \mathbf{R}_{g,l} + \mathbf{R}_{f,l} = \mathbf{D}_l^e\mathbf{R_e}$$

The initial conditions of the local-level position components are chosen as (Bar-Itzhack, 1977):

$$\mathbf{R}_{g,l}(0) = \mathbf{R_l}(0) = \mathbf{D}_l^e\mathbf{R_e}(0) \qquad \mathbf{R}_{f,l}(0) = \mathbf{0} \tag{26}$$

Regarding the initial conditions of the ground velocity components, one recalls from Eq. (7) that:

$$\mathbf{U_l} = \mathbf{V_l} - [\mathbf{\Omega}]_l\mathbf{R_l}$$

and thus, from the initial conditions in Eqs. (19) and (26), the following holds:

$$\mathbf{U}_{f,l}(0) = \mathbf{D}_l^b(0)\mathbf{V}_{f,b}(0) - [\mathbf{\Omega}]_l\mathbf{R}_{f,l}(0) = \mathbf{0}; \qquad \mathbf{U}_{g,l}(0) = \mathbf{U_l}(0) \tag{27}$$

and from Eqs. (26) and (27), Eqs. (24) and (25) are rewritten as:

$$\frac{d\mathbf{U}_{f,l}}{dt^l} = -[2\mathbf{\Omega}+\boldsymbol{\rho}]_l\mathbf{U}_{f,l} + \mathbf{f_l}^{\mathbf{l}}(t^l) \qquad \mathbf{U}_{f,l}(0) = \mathbf{0} \tag{28}$$

$$\frac{d\mathbf{U}_{g,l}}{dt^l} = -[2\mathbf{\Omega}+\boldsymbol{\rho}]_l\mathbf{U}_{g,l} + \mathbf{g_l}(\mathbf{R}(0)) + \mathbf{s}''(h(0)); \qquad \mathbf{U}_{g,l}(0) = \mathbf{U_l}(0) \tag{29}$$

Only $\mathbf{U_{g,l}}$ dynamics is to be considered in the intermediate boundary layer. Because of its slower dynamics relative to $\mathbf{V_{f,b}}$ in the fast boundary layer, the numerical integration of Eq. (29) uses a computation rate T_{int} slower than T_{gyr}.

The algebraic constraint in Eq. (22b) shows that $\mathbf{V_{f,b}}^{\mathbf{l}}(t^l)$ cannot be determined because $[\boldsymbol{\omega}^{\mathbf{bi}}(t^l)]_{\mathbf{b}}$ is singular and, in general, $\mathbf{f}^{\mathbf{l}}(t^l)$ spans the 3-D space. Hence, the asymptotic matching condition seen at the end of the previous Section cannot be met because $\mathbf{V_{f,b}}^{\mathbf{l}}(0)$ cannot be determined from constraint (22b). This is a limitation of casting the navigation problem into an FSP formulation to yield a multirate scheme. It does not imply, however, a limitation of the multirate scheme itself. Rather, it shows that the FSP formulation – suggested here as a theoretical foundation to justify intuitive, appealing approximations used in Bar-Itzhack´s *ad hoc* split b-l multirate scheme, has limitations when applied to this problem. Hence, in practical terms, the violation of the matching condition across boundary layers does not affect the numerical implementation of the multiple time-scale navigation algorithm. As before, the asymptotic matching condition from the intermediate boundary layer to the outer solution is given by:

$$\mathbf{U_l}(0) + \lim_{t^l\to\infty}\{\int_0^{t^l}(-[2\mathbf{\Omega}+\boldsymbol{\rho}]_l\mathbf{U}_{g,l})dt^l + (\mathbf{g_l} + \mathbf{s}'')t^l\} = \mathbf{U}_{g,l}{}^{\mathbf{o}}(0)$$

where $\mathbf{U_{g,l}}^{\mathbf{o}}(0)$ is the outer solution of $\mathbf{U_{g,l}}$ at the onset of the normal time scale.

4.3. The Reduced-Order Problem:

In the present FSP formulation, position $\mathbf{R_e}$ has its dynamics evolving in the normal time scale, and its integration yields the outer solution. From the perturbation of Eqs. (8), (16), (19), and (29), the following is obtained when parameter ε approaches zero:

$$\frac{d\mathbf{R_e}}{dt} = \mathbf{D}_e^b\mathbf{V_b}^{\mathbf{o}}(t) - [\mathbf{\Omega}]_e\mathbf{R_e} = \mathbf{D}_e^l\mathbf{U_l}^{\mathbf{o}}(t); \qquad \mathbf{R_e}(0)\text{ given} \tag{30}$$

$$\mathbf{0} = -[\boldsymbol{\omega}^{\mathbf{bi}}(t)]_{\mathbf{b}}\mathbf{V}_{f,\mathbf{b}}{}^{\mathbf{o}}(t) + \mathbf{f_b}^{\mathbf{o}}(t) \tag{31a}$$

$$\mathbf{0} = -[2\mathbf{\Omega}(t)+\boldsymbol{\rho}(t)]_l\mathbf{U}_{g,l}{}^{\mathbf{o}}(t) + \mathbf{g_l}^{\mathbf{o}}(\mathbf{R_e}(t)) + \mathbf{s}''^{\mathbf{o}}(h(t)) \tag{31b}$$

where $\mathbf{V_{f,b}^o}(t)$ and $\mathbf{U_{g,l}^o}(t)$ represent, respectively, the outer solutions of thrust and ground velocities in the reduced problem. Position dynamics are taken into account here, and thus $\mathbf{g_l^o}(\mathbf{R_e})$ and $\mathbf{s''^o}(h)$ vary in time. However, Equations (31a) and (31b) show that $\mathbf{V_{f,b}^o}$ and $\mathbf{U_{g,l}^o}$ cannot be determined because both $[\boldsymbol{\omega}^{bi}(t)]_{\mathbf{b}}$ and $[2\boldsymbol{\Omega}+\boldsymbol{\rho}(t)]_{\mathbf{l}}$ are singular. Therefore, the above contraints do not determine the outer solution of the fast variables. In other words, in spite of its attractiveness in terms of a theoretical background to multirate navigation algorithms that reduce the computational burden of real-time numerical integration, the FSP formulation produces mathematical constraints that violate asymptotic matching conditions across the boundary layers.

4.4. The discrete-time *modified* b-l split-coordinate multirate algorithm

A closed-loop realization of the forced dynamical separation called for the numerical integration of the navigation dynamics in distinct time scales, with computations sharing the time-varying state values across the boundary layers. Thus, $\mathbf{V_{f,b}}$ dynamics in the fast boundary layer was numerically integrated with the short computation period T_{gyr}, $\mathbf{U_{g,l}}$ dynamics in the intermediate boundary layer with the intermediate period T_{int}, and position dynamics in the outer layer with the large period T_{nav}. The computations in the fast boundary layer were restarted at the end of each computation cycle of the outer solution, with $\mathbf{V_{f,b}}$ reset to its zero initial condition. Moreover, by selecting a sufficiently fast computation rate for the outer solution, $\mathbf{R_{f,l}}$ remains essentially unchanged from its null initial condition seen in Eq.(26), and thus:

$$\mathbf{U_l} = [V_N \;\; V_E \;\; V_D]^T = \mathbf{U_{g,l}} + \mathbf{U_{f,l}} = \mathbf{U_{g,l}} + \mathbf{D_l^b V_{f,b}} - [\boldsymbol{\Omega}]_{\mathbf{l}} \mathbf{R_{f,l}} \approx \mathbf{U_{g,l}} + \mathbf{D_l^b V_{f,b}} \tag{32}$$

Thus, at the end of each T_{nav} computation cycle of the outer solution, $\mathbf{U_{g,l}}$ was reset to the updated $\mathbf{U_l}$, and the computations in the intermediate boundary layer were restarted. Finally, position dynamics given by Eq. (30) was written as in Eq.(9), with $\mathbf{U_l}$ obtained from Eq.(32). Table (1) depicts the discrete-time multirate scheme, termed the *modified* b-l split-coordinate computation scheme after Bar-Itzhack, 1977, with the inclusion of vertical channel damping. The algorithm used an Euler approximation to the derivative operator in each time scale of Eqs.(19), (29), and (9). Consequently, when solving for $\mathbf{V_{f,b}}$, the inertial velocity and angular incremental measurements $\boldsymbol{\Delta\beta_b}$ and $\boldsymbol{\Delta\phi_b}$, respectively, were used in a straightforward manner. In what follows, a discrete-time algorithm is presented that determines the direction cosine matrix $\mathbf{D_l^b}$, which describes the vehicle's attitude relative to the local level.

Table 1 – Discrete-time modified b-l split-coordinate computational scheme

Computation rate	Navigation and attitude equations	Initial conditions	Algebraic relations and reset commands
Fast (T_{gyr})	$\mathbf{V_{f,b}}(k+1) = \mathbf{V_{f,b}}(k) - [\boldsymbol{\Delta\phi}]_{\mathbf{b}}(k)\mathbf{V_{f,b}}(k) + \boldsymbol{\Delta\beta_b}(k)$ (19) Relative quaternion update (33)	$\mathbf{V_{f,b}}(0) = \mathbf{0}$ (19)	$\mathbf{V_{f,b}}(k) \leftarrow \mathbf{0}$ every T_{nav}
Intermediate (T_{int})	$\mathbf{U_{g,l}}(k+1) = \mathbf{U_{g,l}}(k) + T_{int}\{-[2\boldsymbol{\Omega}+\boldsymbol{\rho}]_{\mathbf{l}}(k)\mathbf{U_{g,l}}(k) + \mathbf{g_l}(\mathbf{R}(k)) + \mathbf{s''}(h(k))\}$ (29)	$\mathbf{U_{g,l}}(0) = \mathbf{U_l}(0)$ (27)	$\mathbf{U_{g,l}}(k) \leftarrow \mathbf{U_l}(k)$ every T_{nav}
Slow (T_{nav})	$\lambda(k+1) = \lambda(k) + T_{nav}\{V_N(k)/[R_N(\lambda(k)) + h(k)]$ $\Lambda(k+1) = \Lambda(k) + T_{nav}\{V_E(k)/\{[R_E(\lambda(k)) + h(k)]\cos(\lambda(k))\}$ $h(k+1) = h(k) + T_{nav}\{-V_D(k) + s'(h(k))\}$ (9)	$\lambda(0) = \lambda_0$ $\Lambda(0) = \Lambda_0$ (9) $h(0) = h_0$	DCM update from Eq. (33) $\mathbf{U_l} = \mathbf{U_{g,l}} + \mathbf{D_l^b V_{f,b}}$ (32) $R_N=R_0(1-2e+3e\sin^2(\lambda))$ $R_E=R_0(1+e\sin^2(\lambda))$ $\mathbf{g_l}(\mathbf{R})$ from WGS-84 $\mathbf{s''}(h)=[0\ 0\ (h_{aux}-h)]^T$ $s'(h)=(h-h_{aux})$

5. Attitude Determination: Computation of $\mathbf{D_l^b}$

So far, the computation of the direction cosine matrix (DCM) relating body and local-level frames was assumed to be ideal. Attitude determination is required to transform the specific force measurements from the body frame to the local-level navigation frame. Rate gyro data is often output as sampled incremental angles of body rotation relative to inertial space. These samples are processed here with a relative quaternion discrete-time algorithm derived via an alternative approach (Waldmann, 2002). As described earlier in its continuous-time version (Bar-Itzhack, 1977), it obviates the need to represent the earth angular rate in the body frame. The algorithm is given by the following equations, where $\mathbf{q_b^l}(kT_{gyr}) = (\lambda \;\; \rho_x \;\; \rho_y \;\; \rho_z)^T$ is the ordered quadruple representation of the rotation quaternion that aligns the local-level frame with the body frame:

$$\mathbf{q}_\mathbf{b}^\mathbf{l}((k+1)T_{gyr}) = (\mathbf{I_4} - 1/2[T_{gyr}\boldsymbol{\omega}_\mathbf{l}^\mathbf{li}] + 1/2[T_{gyr}\boldsymbol{\omega}_\mathbf{b}^\mathbf{bi}])\mathbf{q}_\mathbf{b}^\mathbf{l}(kT_{gyr}); \qquad \mathbf{q}_\mathbf{b}^\mathbf{l}(0) \text{ given}$$

$$-[T_{gyr}\boldsymbol{\omega}_\mathbf{l}^\mathbf{li}] = \left[\begin{array}{c|c} 0 & (\Delta\boldsymbol{\phi}_\mathbf{l})^T \\ \hline -\Delta\boldsymbol{\phi}_\mathbf{l} & -\{\Delta\boldsymbol{\phi}_\mathbf{l}\} \end{array}\right]; \quad \Delta\boldsymbol{\phi}_\mathbf{l} = T_{gyr}\left[\Omega\cos(\hat{\lambda}) \;\; 0 \;\; -\Omega\sin(\hat{\lambda})\right]^T;$$

$$\{\Delta\boldsymbol{\phi}_\mathbf{l}\} = T_{gyr}\begin{bmatrix} 0 & (\Omega+\hat{\dot{\Lambda}}(k))\sin(\hat{\lambda}(k)) & -\hat{\dot{\lambda}}(k) \\ -(\Omega+\hat{\dot{\Lambda}}(k))\sin(\hat{\lambda}(k)) & 0 & -(\Omega+\hat{\dot{\Lambda}}(k))\cos(\hat{\lambda}(k)) \\ \hat{\dot{\lambda}}(k) & (\Omega+\hat{\dot{\Lambda}}(k))\cos(\hat{\lambda}(k)) & 0 \end{bmatrix}$$

$$[T_{gyr}\boldsymbol{\omega}_\mathbf{b}^\mathbf{bi}] = \left[\begin{array}{c|c} 0 & -(\Delta\boldsymbol{\phi}_\mathbf{b})^T \\ \hline \Delta\boldsymbol{\phi}_\mathbf{b} & -\{\Delta\boldsymbol{\phi}_\mathbf{b}\} \end{array}\right]; \quad \Delta\boldsymbol{\phi}_\mathbf{b} = \left[\Delta\phi_{bx} \;\; \Delta\phi_{by} \;\; \Delta\phi_{bz}\right]^T; \quad \{\Delta\boldsymbol{\phi}_\mathbf{b}\} = \begin{bmatrix} 0 & -\Delta\phi_{bz} & \Delta\phi_{by} \\ \Delta\phi_{bz} & 0 & -\Delta\phi_{bx} \\ -\Delta\phi_{by} & \Delta\phi_{bx} & 0 \end{bmatrix}; \tag{33}$$

where T_{gyr} is the sampling time associated with the incremental angle samples $\Delta\boldsymbol{\phi}_\mathbf{b}$ provided by the rate gyros, and $\Delta\boldsymbol{\phi}_\mathbf{l}$ is updated at the end of every slow T_{nav}-cycle used in the solution of position dynamics. The DCM transforming from the local-level frame to the body frame was then computed from the rotation quaternion computed with Eq.(33) (Siouris, 1993; Waldmann, 2002).

6. Results

The multirate scheme in Tab. (1) was compared with a simultaneous integration approach shown in Tab. (2). Sensors were assumed to be perfect *because the motivation was to compare the impact of FSP's forced dynamical decoupling of Eqs. (19), (29), and (9) with the simultaneous integration of corresponding Eqs.(14) and (9), the latter approach using the slow T_{nav}-cycle due to limited navigation computer throughput.* Both schemes updated the quaternion solution at the fast T_{gyr}-cycle rate. Two situations were investigated. In the first case, the reference trajectory assumed the vehicle's CM was resting relative to the ground, with the inertial measurement unit (IMU) located 1m ahead of the vehicle's CM. In the second case, the reference trajectory of the CM relative to the ground was:

$$V_N = 200\sin(2\pi t/1200)[m/s] \quad V_E = 150\sin(2\pi t/1200)[m/s] \quad V_D = 5\sin(2\pi t/300)[m/s] \quad t \in [0,300][s]$$

The influence of attitude determination errors caused by angular incremental data was assessed by rotating the vehicle according to a coning motion about its CM to maximize the noncommutativity of finite rotations (Bortz, 1971). The following coning motion relative to the local-level reference frame was used:

$$\boldsymbol{\omega}_\mathbf{b}^\mathbf{bl} = \begin{bmatrix} -\Omega_p\sin(\theta)\cos(\Omega_p t) \\ -\Omega_p\sin(\theta)\sin(\Omega_p t) \\ \Omega_p(\cos(\theta)-1) \end{bmatrix}; \quad \theta = \pi/2[rd]; \quad \Omega_p = \pi/6[rd/s]$$

where θ and Ω_p were the cone half-angle and the precession rate of body axis $\mathbf{z_b}$ about the local vertical, respectively. Inertial data samples $\Delta\boldsymbol{\beta}_\mathbf{b}$ and $\Delta\boldsymbol{\phi}_\mathbf{b}$ were acquired once in every fast computation cycle $T_{gyr}=T_{nav}/N$, and N={12,120} were values used in the simulation. Hence, the fast boundary layer's rate, used to compute the body-frame thrust velocity $\mathbf{V_{f,b}}$ dynamics, matched the inertial data acquisition rate. The intermediate layer, where the gravity-driven local-level ground velocity $\mathbf{U_{g,l}}$ evolves, had its computation cycle given by $T_{int}=T_{nav}/P$, P={4,40}. It is emphasized that $\mathbf{V_{f,b}}$ was reset to zero and $\mathbf{U_{g,l}}$ to the updated $\mathbf{U_l}$ at the end of every slow T_{nav}-cycle, as shown in Tab. (1). In the alternative integration scheme, shown in Tab. (2), Eqs. (14) and (9), and the DCM that resulted from the fast solution of Eq. (33) were simultaneously solved employing the slow T_{nav}-cycle.

Table (3) indicates the root sum of square (RSS) errors in the horizontal plane at the end of the trajectory. The results indicate a significant impact of discretization error on navigation accuracy in comparison with that attained in the continuous-time solution, not shown here due to space limitations. In both discrete-time schemes, the navigation error was attenuated at the expense of a heavier computational burden, i.e., by decreasing T_{nav}. To be effective, it had to be accompanied by a suitable reduction of T_{gyr}, a step analogous to a reduction of the fixed-time step used to numerically solve the continuous-time differential equations. Higher data acquisition rates better captured motion dynamics, and thus improved accuracy. No relevant change in performance resulted from varying the intermediate-cycle period T_{int}. Table (3) shows that for the multirate scheme with a given T_{nav}, accuracy was much improved after the fast cycle T_{gyr} was reduced tenfold. Improving accuracy by means of faster data acquisition, however, is limited by available computer throughput. Hence, it is recommended to keep an adequate *balance* between the reset interval and data acquisition rate for a desired accuracy. In the case of moving vehicle's CM, Table (3) shows significant improvement in the simultaneous approach at the expense of a heavier computational burden with T_{gyr}=8.33×10^{-4}s, when T_{nav}=9.996×10^{-2}s was reduced tenfold to yield an accuracy comparable to that of the multirate scheme.

For a given fast-to-slow cycle ratio that properly captured the dynamics of the motion, the multirate scheme consistently developed less navigation error at the end of the trajectory. This is because incremental velocity data were processed at the fast rate, whereas in the simultaneous approach, on the other hand, a constant specific force was assumed during the slow T_{nav}-cycle.

6. Conclusion

Forced singular perturbations (FSP) were investigated as a theoretical frame to justify a linearized split-frame multiple time scale solution to the strapdown navigation problem. It is motivated by the existence of limited computational resources in autonomous vehicles. Limitations of FSP as a theoretical frame were pointed out in terms of violations of asymptotic matching conditions of the zeroth-order solution across the boundary layers.

Numerical tests were carried out to compare the performance of a variation of the b-l split coordinate multirate computational scheme proposed by Bar-Itzhack with that of a simultaneous approach to the integration of the navigation equations. Both used the same discrete-time relative quaternion update algorithm for attitude determination. Sensors were assumed to be perfect to evaluate the errors caused exclusively by the computational schemes. The results indicated that discretization and use of incremental inertial data had a relevant influence on the attained navigation accuracy. A balanced ratio between the reset interval and the acquisition rate should be selected, while taking into consideration the limited computer throughput, expected motion dynamics, and desired navigation accuracy. The results are valuable in the design of autonomous vehicles expected to accomplish navigation by purely inertial means for brief periods of time.

Table 2 – Discrete-time simultaneous l-coordinate computational scheme

Computation rate	Navigation and attitude equations to solve	Initial conditions	Algebraic relations
Fast (T_{gyr})	Relative quaternion update (33)	$\mathbf{q}_b^l(0)$ given	
Slow (T_{nav})	$\mathbf{U}_l(k+1) = \mathbf{U}_l(k) + T_{nav}\{-[2\boldsymbol{\Omega}+\rho]_l(k)\mathbf{U}_l(k) + $ $+\mathbf{f}_l(k) + \mathbf{g}_l(\mathbf{R}(k)) + \mathbf{s}''(h(k))\}$ (14) $\lambda(k+1) = \lambda(k) + T_{nav}\{V_N(k)/[R_N(\lambda(k)) + h(k)]$ $\Lambda(k+1) = \Lambda(k) + T_{nav}\{V_E(k)/\{[R_E(\lambda(k)) +$ $+h(k)]\cos(\lambda(k))\}$ $h(k+1) = h(k) + T_{nav}\{-V_D(k) + s'(h(k))\}$ (9)	$\mathbf{U}_l(0)$ given $\lambda(0) = \lambda_0$ $\Lambda(0) = \Lambda_0$ $h(0) = h_0$	DCM update from Eq. (33) $\mathbf{f}_l = \mathbf{D}_l^b\mathbf{f}_b$ (15) $\mathbf{f}_b(k) = \Delta\boldsymbol{\beta}_b(nT_{gyr})/T_{gyr}$ at $kT_{nav} = nT_{gyr}$ $R_N=R_0(1-2e+3e\sin^2(\lambda))$ $R_E=R_0(1+e\sin^2(\lambda))$ $\mathbf{g}_l(\mathbf{R})$ from WGS-84 $\mathbf{s}''(h)=[0\ 0\ (h_{aux}-h)]^T$ $s'(h)=(h-h_{aux})$

Table 3 – Navigation error performance with vehicle undergoing coning motion.

	Vehicle's CM at rest		Vehicle's CM in motion		
T_{nav}[s]	9.996×10^{-2}	9.996×10^{-2}	9.996×10^{-2}	9.996×10^{-2}	9.996×10^{-3}
T_{gyr}[s]	8.33×10^{-3} (N=12)	8.33×10^{-4} (N=120)	8.33×10^{-3} (N=12)	8.33×10^{-4} (N=120)	8.33×10^{-4} (N=12)
Simult. RSS[m]	7	1	119	45	11
Mult. RSS[m]	17	1	79	15	9

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8. References

Ardema, M. D., 1976, “Solution of the minimum time-to-climb problem by matched asymptotic expansions,” AIAA Journal, Vol.14, No.7, pp.843-850.

Ardema, M. D., and Rajan, N., 1985a, “Separation of time scales in aircraft trajectories optimization,” Journal of Guidance, Vol.8, No.2, pp.275-278.

Ardema, M. D., and Rajan, N., 1985b, “Slow and fast variables for three-dimensional flight dynamics,” Journal of Guidance, Vol.8, No.4, pp.532-535.

Bar-Itzhack, I. Y., 1977, “Navigation Computation in Terrestrial Strapdown Inertial Navigation Systems,” IEEE Transactions on Aerospace and Electronic Systems, Vol.13, No.6, pp.679-689.

Bar-Itzhack, I. Y., 1978, “Corrections to "Navigation Computation in Terrestrial Strapdown Inertial Navigation Systems",” IEEE Transactions on Aerospace and Electronic Systems, Vol.14, No.3, pp.542-544.

Bar-Itzhack, I. Y. and Berman, N., 1988, “Control Theoretic Approach to Inertial Navigation Systems,” Journal of Guidance, Control, and Dynamics, Vol.11, No.3, pp.237-245.

Bortz, J. E., 1971, “A New Mathematical Formulation for Strapdown Inertial Navigation,” IEEE Transactions on Aerospace and Electronic Systems, Vol.7, No.1, pp.61-66.Kokotovic, P. V., O'Malley, R. E., and Sanutti, P., 1976, “Singular perturbations and order reduction in control theory: an overview,” Automatica, Vol.12, pp.123-132.

Calise, A. J., 1978, “A new boudary layer matching procedure for singularly perturbed systems,” IEEE Transactions on Automatic Control, Vol.23, No.3, pp.510-511.

Calise, A. J., 1981, “Singular perturbation techniques for on-line optimal flight-path control,” Journal of Guidance and Control, Vol.4, No.4, pp.398-405.

Freedman, M. I., and Granoff, B., 1976, “Formal asymptotic solution of a singularly perturbed nonlinear optimal control problem,” Journal of Optimization Theory and Applications, Vol.19, No.2, pp.301-325.

Kelley, H. J., and Edelbaum, T. N., 1970, “Energy climbs, energy turns and asymptotic expansions,” Journal of Aircraft, Vol.7, No.1, pp.93-95.

Kelley, H. J., 1971, “Flight path optimization with multiple scales,” Journal of Aircraft, Vol.8, No.4, pp.238-240.

Kelley, H. J., 1971, “Reduced-order modeling in aircraft mission analysis,” AIAA Journal, Vol.9, No.2, pp.349-350.

Kelley, H. J., 1973, “Aircraft maneuver optimization by reduced-order approximation,” Control and Dynamic Systems, edited by C. T. Leondes, Vol.10, Academic Press, New York, pp.131-178.

Kokotovic, P. V., O'Malley, R. E., and Sanutti, P., 1976, “Singular perturbations and order reduction in control theory: an overview,” Automatica, Vol.12, pp.123-132.

Kokotovic, P. V., Allemong, J. J., Winkelman, J. R., and Chow, J.H., 1980, “Singular perturbation and iterative separation of time scales”, Automatica, Vol.16, pp.23-33.

Shinar, J., 1983, “On applications of singular perturbation techniques in nonlinear optimal control,” Automatica, Vol.19, No.2, pp.203-211.

Shinar, J., 1985, “Zeroth-order feedback strategies for medium range interception in a horizontal plane,” Journal of Guidance, Vol.8, No.1, pp.9-15.

Siouris, G.M., 1993, “Aerospace avionics systems: a modern synthesis,” Academic Press, San Diego.

Sridhar, B., and Gupta, N. K., ,1980, “Missile guidance laws based on singular perturbation methodology,” Journal of Guidance and Control, Vol.3, No.2, pp.158-165.

Tihonov, A. N., 1952, “Systems of differential equations containing small parameters in the derivatives,” Mathematichiskii Sbornik, Vol.31, No.73, pp.575 (in Russian).

Waldmann, J., 2002, “Relative quaternion discrete-time algorithm for terrestrial navigation with strapdown inertial sensors,” Proceedings of the XIV Brazilian Congress on Automation, Natal, Rio Grande do Norte, Brazil, pp.2361-2366.

Wasow, W. R., 1965, “Asymptotic expansion for ordinary differential equations,” Interscience, New York.