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STRESS WAVES IN A MICRO-PERIODIC LAYERED ELASTIC SEMI-SPACE

Józef Ignaczak
Center of Mechanics, Institute of Fundamental Technological Research
Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warsaw, Poland

Abstract. *A one-dimensional initial-boundary value problem of isothermal elastodynamics for a micro-periodic layered semi-space in which a microstructural length is taken into account, is studied. In such a theory, called Refined Averaged Theory (RAT), the layered semi-space is composed of identical two-layer homogeneous isotropic elastic subunits that are mechanically bonded to form a spatially periodic pattern. The field equations consist of: (i) The h-approximation of a displacement field which means that the displacement is a linear function of a micro-periodic shape function h, (ii) Two equations of motion involving a stress S, a body force H, and two coefficients of the h-approximation: a macro-displacement U and a displacement corrector V, and (iii) Two constitutive relations connecting (S,H) to (U,V). By eliminating U,V, and H from (ii) and (iii), one obtains the stress equation of motion for S involving a high frequency parameter. In the paper a pure stress initial-boundary value problem for the micro-periodic layered semi-space is formulated, and a uniqueness theorem for the problem is proved. The proof is based on observation that the stress problem desribed by the partial differential equation subject to suitable initial and boundary conditions can be replaced by the problem involving an integro-differential equation for which an energy integral vanishes. Also, a closed-form Green's function for the pure stress problem is obtained, and a number of properties of the function are revealed.*

Keywords : *Solid Mechanics, Wave Propagation, Composites, Microperiodic Layering*

1. INTRODUCTION

A classical initial-boundary value problem that describes propagation of an elastic wave in a layered semi-space $x \geq 0$ is a problem in which we are to find an elastic process (see Gurtin, 1972, p. 215) that satisfies the field equations of linear elastodynamics inside each of the layers and for every time $t \geq 0$, suitable interface conditions, initial conditions, and boundary conditions at $x = 0$ and $x = \infty$. If the number of layers is very large, then even in the case of a one-dimensional problem in which a periodically layered semi-space is subject to uniform dynamical pressure on its boundary $x = 0$, an analytical solution to the problem is not feasible. For a microperiodic layered semi-space one can replace classical formulation of the problem by an approximate one in which a wave propagates in a semi-space with smeared material properties. The procedure leading to formulation of the approximate problem is based on a linear approximation of the displacement field with respect to a microperiodic shape function $h = h(x)$, and on a quadratic approximation of the associated kinetic and potential energy densities. The associated Hamilton-Kirchhoff principle leads to the governing equations and the stress boundary condition for the approximate problem (cf. C. Woźniak and M. Woźniak, 1995). A characteristic feature of the approximate method is that a microstructure length l that stands for the period of layering, is included in the formulation: in the displacement field equations a coefficient proportional to l^2 appears, and in the stress field equation a characteristic high frequency proportional to l^{-1} appears. By letting $l \to 0$ in the formulation one obtains a problem of the effective modulus theory (EMT) in which the microstructural length is absent. If $l > 0$, the formulation is more refined than that of EMT, and for this reason a theory that accommodates the formulation with $l > 0$ is called a Refined Average Theory (RAT).

In the present paper RAT is used to study a one-dimensional initial-boundary value problem of isothermal elastodynamics for a micro-periodic layered semi-space subject to uniform pressure on its boundary.

In Section 2 the basic equations of RAT for a microperiodic layered semi-space are recalled, and a stress formulation of the problem is presented. In Section 3 a uniqueness theorem for the stress initial-boundary value problem of Section 2 is proved. And in Section 4 a series form of the Green's function for the stress problem is obtained, while in Section 5 final conclusions are presented.

2. BASIC FIELD EQUATIONS FOR A MICROPERIODIC LAYERED ELASTIC SEMI-SPACE

Consider a layered semi-infinite elastic solid composed of an infinite number of identical subunits that are mechanically bonded to form a spatially periodic pattern as shown in Fig.1.

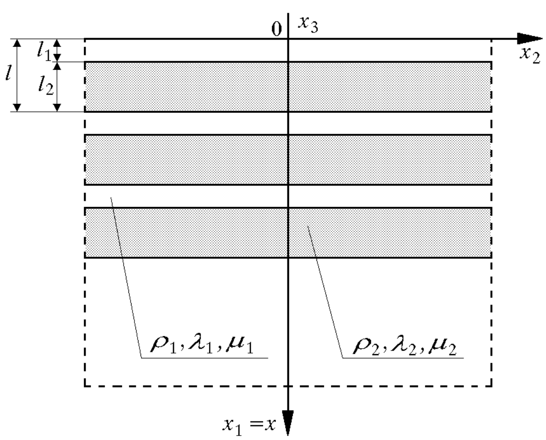

Figure 1- Configuration of a microperiodic layered elastic semi-space.

Each subunit consists of two layers that, in general, have different dimensions and are made of different homogeneous isotropic elastic materials. Let l_i, ρ_i, λ_i and μ_i ($i = 1, 2$), respectively, denote the physical dimension, density, Lamé modulus, and shear modulus of the ith layer in a subunit. If the interface conditions between any two adjacent layers are assumed to be of an ideal mechanical contact type, that is, the displacement and stress vector are continuous across an interface, and a mechanical load is uniformly distributed over the boundary $x = 0$ for every $t \geq 0$, an elastic process in the layered semi-space can be described by a solution to a one-dimensional initial boundary value problem of classical elastodynamics. In such a problem the field equations of homogeneous isotropic elastodynamics are to be satisfied for each layer and suitable initial, interface, and boundary conditions at $x = 0$ and $x = \infty$ are to be met. Since an exact solution to the problem is not feasible, the classical formulation for the layered semi-space is replaced by the approximate one of RAT (see C. Woźniak and M. Woźniak, 1995; and J. Ignaczak, 1999). The field equations of the approximate theory read.

The h-approximation of the displacement field

$$u(x,t) = U(x,t) + h(x)\, V(x,t) \tag{1}$$

The equations of motion

$$\begin{aligned} S_x - <\rho> U_{tt} &= 0 \\ H + <\rho h^2> V_{tt} &= 0 \end{aligned} \tag{2}$$

The constitutive relations

$$\begin{aligned} S &= <\Lambda> U_x + <\Lambda h_x> V \\ H &= <\Lambda h_x> U_x + <\Lambda h_x^2> V \end{aligned} \tag{3}$$

Here, $h = h(x)$ is a dimensionless oscillating periodic funtion on $[0, \infty)$ with period l that satisfies the conditions

$$<h> = 0, \qquad <\eta h> = 0, \qquad <\eta h_x> \neq 0 \tag{4}$$

for any function $\eta = \eta(x)$ on $[0, l]$ of the form

$$\eta(x) = \left\{ \begin{array}{lll} \eta_1 & \text{for} & 0 \leq x < l_1 \\ \eta_2 & \text{for} & l_1 \leq x < l \end{array} \right\} \tag{5}$$

where η_1 and η_2 are constants $(\eta_1 \neq \eta_2)$; and for any function $F = F(x)$ on $[0, l]$ the symbol $< \cdot >$ represents the mean value of F on $[0, l]$

$$<F> = \frac{1}{l} \int_0^l F(x)dx \tag{6}$$

In addition, the function $h = h(x)$ satisfies the asymptotic estimate

$$h(x) = l\ O(1) \qquad \text{as} \qquad l \to 0 \tag{7}$$

If l is small, the function $h = h(x)$ represents a micro-periodic shape function. A typical micro-periodic shape function $h = h(x)$ restricted to the interval $[0, l]$ is shown in Fig. 2.

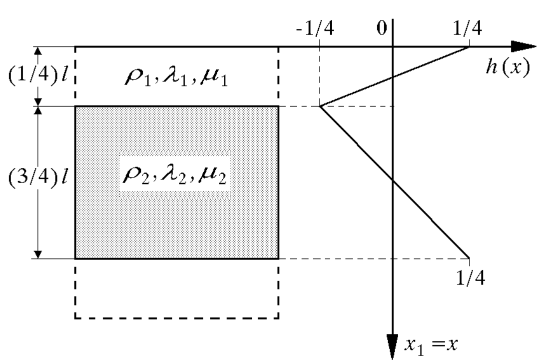

Figure 2- A microperiodic shape function $h = h(x)$ over the interval $0 \leq x < l$.

Other symbols in in Eqs. (1)-(3) have the following meaning. In Eq.(1) the function u represents a displacement in the x-direction; U is a macro-displacement in the x-direction and V is a displacement corrector, respectively. In Eqs. (2)-(3) the function S is a stress component in the x-direction and H is a body force component in the x-direction; moreover, $\Lambda = \lambda + 2\mu$, where λ and μ are Lamé moduli, and ρ is the density. The subscripts in Eqs. (2)-(3) indicate partial derivatives.

The kinetic energy density $K = K(x,t)$ and the potential energy density $P = P(x,t)$, associated with Eqs. (1)-(3), are represented by the functions

$$K(x,t) = \frac{1}{2}\left[< \rho > U_t^2 + < \rho h^2 > V_t^2 \right] \tag{8}$$

and

$$P(x,t) = \frac{1}{2}\left[< \Lambda > U_x^2 + 2 < \Lambda h_x > U_x V + < \Lambda h_x^2 > V^2 \right] \tag{9}$$

Note that Eqs. (2)-(3) form a complete set of four field equations of the one-dimensional theory for the four unknowns: $U, V; S,$ and H. By eliminating the pair (S,H) from Eqs. (2)-(3) we arrive at the displacement field equations for the pair (U,V)

$$\begin{aligned} < \Lambda > U_{xx} + < \Lambda h_x > V_x - < \rho > U_{tt} &= 0 \\ < \Lambda h_x > U_x + < \Lambda h_x^2 > V + < \rho h^2 > V_{tt} &= 0 \end{aligned} \tag{10}$$

Also note that an alternative form of the constitutive relations (3) reads

$$\begin{aligned} U_x &= \frac{1}{< \Lambda^* >}\left[S - \frac{< \Lambda h_x >}{< \Lambda h_x^2 >} H \right] \\ V &= -\frac{1}{< \Lambda^* >}\frac{< \Lambda h_x >}{< \Lambda h_x^2 >}\left[S - \frac{< \Lambda >}{< \Lambda h_x >} H \right] \end{aligned} \tag{11}$$

where

$$< \Lambda^* > = < \Lambda > - \frac{< \Lambda h_x >^2}{< \Lambda h_x^2 >} \qquad (< \Lambda^* > > 0) \tag{12}$$

Therefore, by eliminating the pair (U,V) from Eqs. (2) and (11) we obtain the field equations for the pair (S,H):

$$\begin{aligned} S_{xx} - \frac{< \rho >}{< \Lambda^* >}\left[S_{tt} - \frac{< \Lambda h_x >}{< \Lambda h_x^2 >} H_{tt} \right] &= 0 \\ H - \frac{< \rho h^2 >}{< \Lambda^* >}\frac{< \Lambda h_x >}{< \Lambda h_x^2 >}\left[S_{tt} - \frac{< \Lambda >}{< \Lambda h_x >} H_{tt} \right] &= 0 \end{aligned} \tag{13}$$

Finally, by eliminating the function H from Eqs. (13) we arrive at the stress equation of motion for S:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{< c_1 >^2}\frac{\partial^2}{\partial t^2} \right)\left(\frac{\partial^2}{\partial t^2} + \kappa^2 \right) S - \frac{\omega^2}{< c_1 >^2}\frac{\partial^2}{\partial t^2} S = 0 \tag{14}$$

where

$$\begin{aligned} < c_1 > &= \frac{< \Lambda >^{1/2}}{< \rho >^{1/2}} & \kappa &= \Omega \frac{< \Lambda^* >^{1/2}}{< \Lambda >^{1/2}} \\ \omega &= \Omega \left(1 - \frac{< \Lambda^* >}{< \Lambda >} \right)^{1/2} & \Omega &= \frac{< \Lambda h_x^2 >^{1/2}}{< \rho h^2 >^{1/2}} \end{aligned} \tag{15}$$

Clearly, $< c_1 >$ has the dimension of velocity, and Ω has the dimension of frequency, *i.e.*

$$[< c_1 >] = \left[L\ T^{-1} \right] \qquad [\Omega] = [\kappa] = [\omega] = \left[T^{-1} \right] \tag{16}$$

where L and T stand for the length and time units, respectively; and $[\cdot]$ represents the dimension of a physical quantity.

As a result, for the microperiodic layered semi-space subject to homogeneous initial conditions and uniform pressure $s = s(t)$ on its boundary $x = 0$, the following pure stress initial-boundary value problem may be formulated. Find a stress field $S = S(x,t)$ on $[0, \infty) \times [0, \infty)$ that satisfies the field equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{< c_1 >^2} \frac{\partial^2}{\partial t^2} \right) \left(\frac{\partial^2}{\partial t^2} + \kappa^2 \right) S - \frac{\omega^2}{< c_1 >^2} \frac{\partial^2}{\partial t^2} S = 0 \qquad \text{for} \quad x \geq 0,\ t \geq 0 \tag{17}$$

subject to the initial conditions

$$S(x,0) = 0 \quad \frac{\partial}{\partial t} S(x,0) = 0 \quad \frac{\partial^2}{\partial t^2} S(x,0) = 0 \quad \frac{\partial^3}{\partial t^3} S(x,0) = 0 \qquad \text{for} \quad x \geq 0 \tag{18}$$

and the boundary condition

$$S(0,t) = -s(t) \qquad \text{for} \quad t \geq 0 \tag{19}$$

where $s = s(t)$ is a prescribed function. Moreover, the function S and its partial derivatives of a finite order are to vanish as $x \to \infty$ for every $t > 0$.

If a solution S to the problem (17)-(19) is found, the functions H, U, and V are computed from the formulas

$$H(x,t) = \frac{< \Lambda h_x >}{< \Lambda >} \int_0^t \cos \kappa (t - \tau) \frac{\partial S}{\partial \tau}(x, \tau) d\tau \tag{20}$$

$$U(x,t) = \frac{1}{< \rho >} \int_0^t (t - \tau) \frac{\partial S}{\partial x}(x, \tau) d\tau \tag{21}$$

and

$$V(x,t) = - \frac{1}{< \rho h^2 >} \int_0^t (t - \tau) H(x, \tau) d\tau \tag{22}$$

Also, note that the pure stress problem (17)-(19) contains two high frequency parameters κ and ω, since, by Eqs. (7) and (15) we have

$$\kappa = l^{-1}\ O(1), \qquad \omega = l^{-1}\ O(1), \qquad \text{as} \qquad l \to 0 \tag{23}$$

3. UNIQUENESS THEOREM FOR THE PURE STRESS PROBLEM

Theorem 3.1 The pure stress initial-boundary value problem (17)-(19) may have at most one solution.

Proof is based on observation that the problem (17) - (19) with $s \equiv 0$ is equivalent to the following one. Find a field $\Sigma = \Sigma(x,t)$ on $[0, \infty) \times [0, \infty)$ that satisfies the integro-differential equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{< c_1 >^2}\frac{\partial^2}{\partial t^2}\right)\Sigma - \frac{\omega^2}{< c_1 >^2}\int_0^t \cos\kappa(t-\tau)\,\frac{\partial\Sigma}{\partial\tau}(x,\tau)\,d\tau = 0 \qquad \text{for} \quad x \geq 0,\ t \geq 0 \tag{24}$$

subject to the conditions $(\cdot = \partial/\partial t)$

$$\Sigma(x,0) = 0, \qquad \dot{\Sigma}(x,0) = 0 \qquad \text{for} \quad x \geq 0 \tag{25}$$

and

$$\Sigma(0,t) = 0 \qquad \text{for} \quad t \geq 0 \tag{26}$$

Moreover, the function Σ and its partial derivatives of a finite order are to vanish as $x \to \infty$ for every $t > 0$.

A stress field S that satisfies Eqs. (17)-(19) with $s \equiv 0$ is related to Σ by

$$S(x,t) = \frac{1}{\kappa}\int_0^t \sin\kappa(t-\tau)\,\Sigma(x,\tau)d\tau \tag{27}$$

Clearly, if $\Sigma \equiv 0$ on $[0,\infty)\times[0,\infty)$, then by Eq. (27), we obtain $S \equiv 0$ on $[0,\infty)\times[0,\infty)$.

To show that Eqs. (24)-(26) imply that $\Sigma \equiv 0$ on $[0,\infty)\times[0,\infty)$, we multiply Eq. (24) by $\dot{\Sigma}$ and obtain

$$\dot{\Sigma}\Sigma_{xx} - \frac{1}{< c_1 >^2}\dot{\Sigma}\ddot{\Sigma} - \frac{\omega^2}{< c_1 >^2}\dot{\Sigma}(x,t)\int_0^t \cos\kappa(t-\tau)\,\dot{\Sigma}(x,\tau)\,d\tau = 0 \tag{28}$$

Since

$$\dot{\Sigma}\Sigma_{xx} - \frac{1}{< c_1 >^2}\dot{\Sigma}\ddot{\Sigma} = \left(\dot{\Sigma}\Sigma_x\right)_x - \frac{1}{2}\frac{\partial}{\partial t}\left[(\Sigma_x)^2 - \frac{1}{< c_1 >^2}\left(\dot{\Sigma}\right)^2\right] \tag{29}$$

therefore, integrating Eq. (28) over the space-time domain: $0 \leq x < \infty,\ 0 \leq s \leq t$, and using the homogeneous initial-boundary conditions (25)-(26), we obtain

$$\begin{aligned}&\int_0^\infty \left\{\frac{1}{2}\left[\Sigma_x(x,t)\right]^2 + \frac{1}{2}\frac{1}{< c_1 >^2}\left[\dot{\Sigma}(x,t)\right]^2\right\}dx + \\ &\frac{\omega^2}{< c_1 >^2}\int_0^\infty\left\{\int_0^t\int_0^s \dot{\Sigma}(x,s)\,\dot{\Sigma}(x,\tau)\,\cos\kappa(s-\tau)\,d\tau\,ds\right\}dx = 0\end{aligned} \tag{30}$$

Hence, we get the estimate

$$\frac{1}{2}\int_0^\infty \left[\dot{\Sigma}(x,t)\right]^2 dx \leq \omega^2\int_0^\infty\left\{\int_0^t\int_0^s |\dot{\Sigma}(x,s)|\,|\dot{\Sigma}(x,\tau)|\,d\tau\,ds\right\}dx \tag{31}$$

Next, since for an arbitrary function $A = A(t)$ on $[0,\infty)$

$$\int_0^t\left\{\int_0^s |A(\tau)|d\tau\right\}|A(s)|ds = \frac{1}{2}\left(\int_0^t |A(s)|ds\right)^2 \tag{32}$$

therefore, by the inequality (31), we obtain

$$\int_0^\infty \left[\dot{\Sigma}(x,t)\right]^2 \, dx \le \omega^2 \int_0^\infty \left(\int_0^t |\dot{\Sigma}(x,s)| \, ds\right)^2 dx \tag{33}$$

Now, by virtue of the Schwartz inequality, for an arbitrary function $f = f(s)$ on $[0,\infty)$, we get

$$\left(\int_0^t |f(s)| ds\right)^2 \le t \int_0^t |f(s)|^2 ds \tag{34}$$

Hence, by the inequalities (33)-(34), we obtain

$$\int_0^\infty \left[\dot{\Sigma}(x,t)\right]^2 \, dx \le \omega^2 \, t \int_0^\infty \left\{\int_0^t \left[\dot{\Sigma}(x,s)\right]^2 \, ds\right\} dx \tag{35}$$

Therefore, if we introduce the function

$$N(t) = \int_0^\infty \left\{\int_0^t \left[\dot{\Sigma}(x,s)\right]^2 \, ds\right\} dx \tag{36}$$

we check that $N(0) = 0$, and

$$\frac{d}{dt}\left[\exp\left(-\frac{\omega^2 t^2}{2}\right) N(t)\right] \le 0 \tag{37}$$

By integrating this inequality over the interval $[0,t]$, we obtain: $N(t) \equiv 0$ on $[0,\infty)$. This together with Eqs. (25) and (36) imply that $\Sigma \equiv 0$ and $S(x,t) \equiv 0$ on $[0,\infty) \times [0,\infty)$. This completes proof of Theorem 3.1.

4. CLOSED-FORM GREEN'S FUNCTION FOR THE PURE STRESS PROBLEM

One can show that a solution to the problem (17)-(19) takes the form

$$S(x,t) = \frac{1}{\kappa} \int_0^t \sin \kappa(t-\tau) \, \Sigma(x,\tau) d\tau \tag{38}$$

where

$$\Sigma(x,t) = \int_0^t G(x,t-\tau) \left[\ddot{s}(\tau) + \kappa^2 s(\tau)\right] d\tau \tag{39}$$

and $G = G(x,t)$ is a Green's function for the integro-differential equation (24). The function G satisfies the equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{< c_1 >^2}\frac{\partial^2}{\partial t^2}\right) G - \frac{\omega^2}{< c_1 >^2} \int_0^t \cos \kappa(t-\tau) \, \frac{\partial G}{\partial \tau}(x,\tau) \, d\tau = 0 \quad \text{for} \quad x \ge 0, \ t \ge 0 \tag{40}$$

the initial conditions

$$G(x,0) = 0 \qquad \dot{G}(x,0) = 0 \qquad \text{for} \quad x \ge 0 \tag{41}$$

the boundary condition

$$G(0,t) = - \, \delta(t) \tag{42}$$

and vanishing conditions at infinity. In Eq. (42) $\delta = \delta(t)$ is the Dirac delta function.

Using a Laplace transform technique (Ignaczak, 1999) the following series solution of the problem (40)-(42) is obtained

$$G(x,t) = -\delta(t - \frac{x}{< c_1 >}) - H(t - \frac{x}{< c_1 >})\, g\left(x, t - \frac{x}{< c_1 >}\right) \tag{43}$$

where $H = H(t)$ is the Heaviside function

$$H(t) = \left\{ \begin{array}{lll} 1 & \text{for} & t > 0 \\ 0 & \text{for} & t < 0 \end{array} \right\} \tag{44}$$

and $g = g(x,t)$ is the series of Neumann's type for the integro-differential equation (40)

$$\begin{aligned} g(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x\omega^2}{2 < c_1 >}\right)^n \Bigg[\{\cos \kappa t\}^n \; + \\ n \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!} \left(\frac{\omega^2}{4}\right)^\nu \frac{(n+2\nu-1)!}{(n+\nu)!(\nu-1)!} \{t^{\nu-1}\}\{\cos \kappa t\}^{n+\nu} \Bigg] \end{aligned} \tag{45}$$

$$\text{for} \quad x \geq 0, \; t \geq 0$$

Here, for arbitrary functions $a = a(t)$ and $b = b(t)$ on $[0, \infty)$, the symbol $\{a\}\{b\}$ represents the convolution product of $a = a(t)$ and $b = b(t)$ defined by (Mikusiński, 1959)

$$\{a\}\{b\} = \int_0^t a(t-\tau) b(\tau) d\tau \equiv a * b \tag{46}$$

In particular, $\{f\}^n$ represents n-th convolutional power of a function $\{f(t)\} \equiv f(t)$.

With regard to the series (45) the following theorem holds true.

Theorem 4.1 (i) The series (45) and its partial derivatives with respect to x and t of a finite order converge uniformly for every point $(x,t) \in [0,\infty) \times [0,\infty)$ and for arbitrary positive finite parameters ω, κ and $< c_1 >$. (ii) For the function $g = g(x,t)$, represented by the series (45), the pointwise estimate holds true

$$|g(x,t)| \leq \frac{x\omega^2}{2 < c_1 >} \exp\left[\frac{x\omega^2 t}{2 < c_1 >}\right] \exp\left[\frac{\omega^2 t^2}{4}\right] \tag{47}$$

$$\text{for} \quad x > 0, \; t > 0; \; \omega > 0$$

Proof is based on the relations:

(A)

$$|\{\cos \kappa\, t\}^n| \leq \{1\}^n \qquad \text{for} \quad n \geq 1 \tag{48}$$

where

$$\{1\}^n = \frac{1}{(n-1)!}\{t^{n-1}\} \qquad \text{for} \quad n \geq 1 \tag{49}$$

(B)

$$\frac{1}{(\nu-1)!}|\{t^{\nu-1}\}\{\cos \kappa t\}^{n+\nu}| \leq \frac{\{t^{n+2\nu-1}\}}{(n+\nu-1)!\; \nu!} \qquad \text{for} \quad n \geq 1, \; \nu \geq 1 \tag{50}$$

(C)

$$\frac{(n+2\nu-1)!}{(n+\nu-1)!\,(n+\nu)!\,\nu!} \le 1 \qquad \text{for} \quad n \ge 1,\ \nu \ge 1 \tag{51}$$

To prove (A) we use the definition of convolution of two functions and a method of mathematical induction. Similar tools may be used to obtain proof of (B) and (C).

To show (i) we introduce the functions

$$S_1(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x\omega^2}{2 < c_1 >} \right)^n \{\cos \kappa t\}^n \tag{52}$$

$$S_2(x,t) =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \left(\frac{x\omega^2}{2 < c_1 >} \right)^n \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!} \left(\frac{\omega^2}{4} \right)^\nu \frac{(n+2\nu-1)!}{(n+\nu)!(\nu-1)!} \{t^{\nu-1}\}\{\cos \kappa t\}^{n+\nu} \tag{53}$$

Using the inequalities (48) and (50) we obtain

$$|S_1(x,t)| \le \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{x\omega^2}{2 < c_1 >} \right)^n \frac{t^{n-1}}{(n-1)!} \tag{54}$$

and

$$|S_2(x,t)| \le$$

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{x\omega^2}{2 < c_1 >} \right)^n \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left(\frac{\omega^2}{4} \right)^\nu \frac{(n+2\nu-1)!}{(n+\nu)!} \frac{t^{n+2\nu-1}}{(n+\nu-1)!\,\nu!} \tag{55}$$

Next, since $n! \ge 1$ for $n \ge 1$, by Eq. (54), we get

$$|S_1(x,t)| \le \sum_{n=1}^{\infty} \left(\frac{x\omega^2}{2 < c_1 >} \right)^n \frac{t^{n-1}}{(n-1)!} = \frac{x\omega^2}{2 < c_1 >} \exp\left[\frac{x\omega^2 t}{2 < c_1 >} \right] \tag{56}$$

Similarly, by Eqs. (51) and (55), we obtain

$$\begin{aligned} |S_2(x,t)| &\le \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{x\omega^2}{2 < c_1 >} \right)^n t^{n-1} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left(\frac{\omega^2\ t^2}{4} \right)^\nu = \\ &\frac{x\ \omega^2}{2 < c_1 >} \exp\left[\frac{x\ \omega^2\ t}{2 < c_1 >} \right] \left\{ \exp\left[\frac{\omega^2\ t^2}{4} \right] - 1 \right\} \end{aligned} \tag{57}$$

Since

$$g(x,t) = S_1(x,t) + S_2(x,t) \tag{58}$$

therefore, by Eqs. (56), (57), the series (45) converges uniformly for every $(x,t) \in [0,\infty) \times [0,\infty)$ and for arbitrary positive parameters ω, κ and $< c_1 >$. To prove that a derivative of the series (45) with respect to x and t converges uniformly over the same range of variables $(x,t;\omega,\kappa,< c_1 >)$, we proceed in a way similar to that of the series (45). This completes proof of part (i). Finally, by virtue of the inequalities (56), (57) and Eq. (58), we arrive at the pointwise estimate (47). This completes proof of Theorem 4.1.

Remark
The k-th convolutional power of the function $\{\cos \kappa t\}$ that occurs in the series (45) is obtained from the recurrence relation

$$\{\cos \kappa t\}^{k+1} = -\frac{1}{2\ k\ \kappa}\frac{\partial^2}{\partial\kappa\partial t}\{\cos \kappa t\}^k \qquad (k \geq 1) \tag{59}$$

For example, letting $k = 1, 2, 3, 4$ in Eq. (59) we get

$$\{\cos \kappa t\}^2 = \left\{\frac{t}{2}\left(\cos \kappa t\ +\ \frac{\sin \kappa t}{\kappa\ t}\right)\right\} \tag{60}$$

$$\{\cos \kappa t\}^3 = \left\{\frac{t^2}{2!\ 2^2}\left(\cos \kappa t\ +\ 3\ \frac{\sin \kappa t}{\kappa\ t}\right)\right\} \tag{61}$$

$$\{\cos \kappa t\}^4 = \left\{\frac{t^3}{3!\ 2^3}\left[\left(\cos \kappa t\ +\ 6\ \frac{\sin \kappa t}{\kappa\ t}\right) - \frac{3}{\kappa^2 t^2}\left(\cos \kappa t\ -\ \frac{\sin \kappa t}{\kappa\ t}\right)\right]\right\} \tag{62}$$

$$\{\cos \kappa t\}^5 = \left\{\frac{t^4}{4!\ 2^4}\left[\left(\cos \kappa t\ +\ 10\ \frac{\sin \kappa t}{\kappa\ t}\right) - \frac{15}{\kappa^2 t^2}\left(\cos \kappa t\ -\ \frac{\sin \kappa t}{\kappa\ t}\right)\right]\right\} \tag{63}$$

5. CONCLUSIONS

1. Modeling of transient stress waves in a microperiodic layered elastic semi-space using RAT amounts to the study of an integro-differential equation rather than to that of a partial differential equation.

2. The integro-differential equation subject to suitable initial and boundary conditions may have at most one solution.

3. The stress wave generated by a solution to the integro-differential problem may be represented in the form of a Neumann's series associated with the integro-differential operator.

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