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STATISTICAL ANALYSIS OF A CONTROLLED MOTION

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Abstract. *A technique developed for statistical analysis of complex controlled dynamical systems operating in presence of random disturbances is presented in this paper. The main peculiarity of the technique consists in the estimation not only moments (mean, variance, covariance) of output variables but also the probability and the quantile as system accuracy criteria. Use of that criteria allows to utilize more convenient (in comparison with traditional ones) system performance characteristics, as well as to reduce computation time, necessary for solving the problem by the Monte Carlo simulation method.*

Keywords: *Launcher, Random disturbances, Linearization, Probabilistic analysis*

1. INTRODUCTION

A satellite is one of example of high precise dynamical system. It should be inserted into an orbit with high accuracy. This accuracy has to be provided by a launcher controlled motion in presence of numerous random factors: initial conditions, vehicle parameters, and external disturbances acting upon a vehicle during a flight. Impact of those random factors is evaluated using priory statistical analysis methods. Usually covariance analysis is conducted providing estimation of moments (mean, variances and covariance moments) of state variables of a vehicle along a trajectory using analytical methods or the Monte Carlo simulation approach [Malyshev at all, 1996]. But in many cases it is practically useful to evaluate probability of an event, that a state vector or its some components are within a given (admissible) area, or a size of the area, within which considered state variables are with given probability (quantile). Evaluation of a probability and (or) quantile is called as a probabilistic analysis of a system [Malyshev, Kibzun, 1987]. A purpose of this paper is presentation of a technique developed for effective solution of a probabilistic analysis problem more convenient and less time consuming in comparison with existing techniques. The technique description is given here in application to a VLS-type solid-propellant launch vehicle intended for satellite injection into the circular orbit.

2. PROBLEM STATEMENT OF A PROBABILISTIC ANALYSIS

A general problem of a probabilistic analysis of a dynamic system is presented in application to a launch vehicle trajectory analysis. A probabilistic analysis of a launcher motion is performed here in respect to current position, current velocity and terminal accuracy of a satellite insertion into an orbit.

Probabilistic analysis of a launcher 3D current position. A sphere as an admissible area for current deviations of launcher coordinates from nominal ones is considered. A module of a vehicle deviation from nominal position in 3D motion at a current moment t can be characterized by the scalar variable $\hat{O}_1(t)$:

$$\Phi_1(t) = \sqrt{(x(t) - x_n(t))^2 + (y(t) - y_n(t))^2 + (z(t) - z_n(t))^2}\,, \qquad (1)$$

where $x_n(t), y_n(t), z_n(t)$ are coordinates of a launcher at a nominal trajectory. The variable $\hat{O}_1(t)$ is random function of time due to random disturbances acting upon a vehicle. The probabilistic analysis of a launcher current position consists in a determination of a probability $\mathrm{P}\{\Phi_1(t) \le \varphi_1\}$, where φ_1 is an admissible level of the position accuracy.

Probabilistic analysis of a launcher current velocity. A sphere as an admissible area for current deviations of launcher velocity components from nominal ones also can be considered. A velocity vector deviation from nominal can be described by the scalar variable $\hat{O}_2(t)$:

$$\hat{O}_2(t) = \sqrt{(V_x(t) - V_{xn}(t))^2 + (V_y(t) - V_{yn}(t))^2 + (V_z(t) - V_{zn}(t))^2}\,, \qquad (2)$$

where $V_{xn}(t), V_{yn}(t), V_{zn}(t)$ are components of a launcher current velocity along the nominal trajectory.

By analogy with the previous case, the objective of a probabilistic analysis of a launcher current velocity is a determination of a probability $\mathrm{P}\{\Phi_2(t) \le \varphi_2\}$, where φ_2 is admissible value of a disturbed velocity vector deviation from nominal one.

Probabilistic analysis of a terminal accuracy for a satellite insertion into orbit. As an example, let us consider a preset admissible terminal 3D area in coordinates: apogee, perigee and inclination of a satellite orbit. In respect to that variables final random state of a launcher and simultaneously orbit of a satellite can be described by the variable $\hat{O}_3(T)$. That variable is characterized by the following properties: $\hat{O}_3(T) = 0$, if the satellite is on the nominal orbit and $\hat{O}_3(T) = 1$, if the satellite state vector is on the border of the admissible area. The inequality $0 < \hat{O}_3(T) < 1$ takes place inside of the admissible area. The variable $\hat{O}_3(T)$ is random variable due to random disturbances. The probabilistic analysis of the terminal accuracy consists in determination of a probability $P\{\hat{O}_3(t) \le 1\}$.

Generalizing for all three cases mentioned above, the probabilistic analysis of launcher motion has as an aim to determine the probabilities:

$$P_{\varphi i}(t) = P\{\hat{O}_i(t) \le \varphi_i\},\ i = \overline{1,3}, \qquad (3)$$

where $\varphi_i, i = \overline{1,3}$ are preset values of scalar performance indexes $\Phi_i(t)$.

Additionally to the direct probabilistic problem described above, it will be discussed the inverse probabilistic problems in which a preset level of probability α is supposed, i.e. the inequality $P_{\varphi i}(t) \ge \alpha$ has to be satisfied. In this case it is necessary to determine minimum value of a parameter φ_i (quantile), with which mentioned inequality is valid. In another words the inverse problem is a problem of the evaluation of the best accuracy provided with a preset probability. Mathematically the quantile determination problem is formulated as follows:

$$\hat{O}_{\alpha i}(t) = \min\{\varphi : P_{\varphi_i}(t) \ge \alpha\}, i = \overline{1,3}. \tag{4}$$

Main phases of a technique for solving the direct and inverse probabilistic analysis problems are presented below.

3. MATHEMATICAL MODEL FOR LAUNCHER

A mathematical model of a launcher motion in presence of random disturbances has to be written in the Cauchy form:

$$\dot{\bar{z}} = f(\bar{z}, \omega, t), \quad \bar{z}(0) = \bar{z}_0, \tag{5}$$

where $\bar{z} = col(V_x, V_y, V_z, x, y, z)$ is a state vector, $\omega = \mathrm{col}(\omega_1, \ldots \omega_N)$ is vector of dimension N, consisting off all disturbances – random variables acting upon a launcher during a flight. Formally, the following differential equations can be written for components of that vector:

$$\dot{\omega}_i = 0, \quad i = \overline{1, N}. \tag{6}$$

Jointly Eq. (5) and (6) form the augmented system:

$$\left.\begin{array}{l} \dot{\bar{z}} = f(\bar{z}, \omega, t), \bar{z}(0) = \bar{z}_0, \\ \dot{\omega} = 0, \omega(0) = \omega_0, \end{array}\right\} \tag{7}$$

or in the vector form:

$$\dot{z} = f(z, t), \qquad z(0) = z_0, \tag{8}$$

where $z = col(\bar{z}, \omega)$ is an augmented state vector of a system.

4. LINEARIZATION

Further, linearized equations of disturbed motion of a vehicle are used. An ordinary linearization and a statistical linearization are used. The first one (ordinary linearization) is utilized in respect to "smooth" nonlinear functions of a system and the second one – in respect to "substantial" nonlinear functions.

Let $\tilde{z}(t)$ is a state vector consisting of state variables of a launcher mass center position *X, Y, Z* and velocity V_x, V_y, V_z and vector $\tilde{\omega} = \mathrm{col}(\omega_S, \ldots \omega_N)$, consisting of random parameters, in respect to which nonlinear functions in the system (8) are “smooth”. The vector $\tilde{z}(t)$ is described by the equation:

$$\dot{\tilde{z}} = \tilde{f}(\tilde{z}, t), \qquad \tilde{z}(0) = \tilde{z}_0. \tag{9}$$

The right part of Eq (9) can be represented as Taylor’s series in a vicinity of a nominal trajectory. Let $\tilde{z}_n(t)$ is a state vector along a nominal trajectory. The following linear equation describes behavior of the deviation state vector $\Delta\tilde{z}(t) = \tilde{z}(t) - \tilde{z}_n(t)$:

$$\Delta\dot{\tilde{z}}(t) = \frac{\partial \tilde{f}(\tilde{z}, t)}{\partial \tilde{z}} \Delta\tilde{z}(t). \tag{10}$$

Designating

$$\frac{\partial f(\tilde{z}, t)}{\partial \tilde{z}} = \tilde{A}(t), \tag{11}$$

the linearized mathematical model of a launcher disturbed motion is as follows:

$$\Delta\dot{\tilde{z}}(t) = \tilde{A}(t)\Delta\tilde{z}(t), \Delta\tilde{z}(0) = \Delta\tilde{z}_0. \tag{12}$$

5. STATISTICAL LINEARIZATION

More accurate solution can be provided using statistical linearization technique for “substantial” random disturbances $\omega_i, i = \overline{1, S}$ in the right part of Eq. (8). One of these type disturbances is a step change of a vehicle mass during a stage separation.

The statistical linearization method of the nonlinear function $\Delta f_{ij} = \left(f_i(z + \Delta z_j) - f_i(z)\right)$, consists in its approximate replacement by a linear function $a_{ij}\Delta z_j$, which is statistically equal to initial function (Malyshev, Kibzun, 1987):

$$D[\Delta f_{ij}] = D[a_{ij}\Delta z_j] \tag{13}$$

and

$$M\lfloor \Delta f_{ij} \rfloor = M\lfloor a_{ij}\Delta z_j \rfloor, \tag{14}$$

where $f_i(\cdot)$ is i-th element of the vector function $f(\cdot)$; Δz_j is j-th component of the vector z variation due to random disturbances.

The coefficient a_{ij} is estimated using the formula:

$$\left|a_{ij}\right| = \left\{\frac{D\left[\Delta f_{ij}\right]}{D\left[\Delta z_j\right]}\right\}^{\frac{1}{2}} = \left\{\frac{1}{D\left[\Delta z_j\right]}\left[\int_{-\infty}^{\infty}\left(\Delta f_{ij}\right)^2 p(z)dz\right]\right\}. \tag{15}$$

The following expression can be used for the approximate integral calculation in Eq. (15):

$$\left|a_{ij}\right| = \left\{ \frac{1}{D[\Delta z_j]} \left[\sum_{k=1}^{L} \left(\frac{f_i(z+\Delta z_{jk},t) - f_i(z,t)}{\Delta z_{jk}} \Delta z_{jk} \right)^2 p(z_k) S \right] \right\}^{1/2} =$$

$$= \left\{ \frac{1}{D[\Delta z_j]} \left[\sum_{k=1}^{L} \left(f_i(z+\Delta z_{jk}) - f_i(z) \right)^2 P_k \right] \right\}^{1/2}, \quad (16)$$

where P_k are weighting coefficients.

An additional multiplier, that takes into account a sign of a coefficient, is introduced. Then

$$a_{ij} = \operatorname{sign}(f_i(z+\Delta z_{jL},t) - f_i(z,t)) \times$$

$$\times \left\{ \frac{1}{D[\Delta z_j]} \left[\sum_{k=1}^{L} (f_i(z+\Delta z_{jk},t) - f_i(z,t))^2 P_k \right] \right\}^{\frac{1}{2}}. \quad (17)$$

In the specific case, when $D[\Delta z_j] = 1$ the coefficient is equal to:

$$a_{ij} = \operatorname{sign}(f_i(z+\Delta z_{jL},t) - f_i(z,t)) \times$$

$$\times \left[\sum_{k=1}^{L} (f_i(z+\Delta z_{jk},t) - f_i(z,t))^2 P_k \right]^{\frac{1}{2}}. \quad (18)$$

6. AUGMENTED LINEARIZED MODEL

A generalized linearized mathematical model for a disturbed motion of a launcher using both an ordinary linearized model (12) and a statistical linearized model (18) can be written:

$$\Delta\dot{z}(t) = A(t)\Delta z(t), \Delta z(0) = \Delta z_0. \quad (19)$$

Here $A(t)$ is a matrix with elements $a_{ij}, i = \overline{1,6+N}, j = \overline{1,6+N}$. The elements $a_{ij}, i = \overline{1,6}, j = \overline{7,6+S}$ are calculated by (18) and $a_{ij} = \tilde{a}_{ij}, i = \overline{1,6}, j = \overline{1,6}$, $a_{i,j+4} = \tilde{a}_{ij}, i = \overline{1,6}, j = \overline{7,6+N-S}$, $a_{ij} = 0, i = \overline{7,6+N}, j = \overline{1,6+N}$, where $\tilde{a}_{ij}, i = \overline{1,6}, j = \overline{1,6+N-S}$ are elements of matrix $\tilde{A}(t)$.

7. COVARIANCE ANALYSIS

The purpose of covariance analysis is to determine a covariance matrix $K_{\Delta z}(t)$ of a vector $\Delta z(t)$.

According to the formed model of a disturbed motion the elements k_{ii}, $i = \overline{7,6+N}$ of covariance matrix $K_{\Delta z}(t)$ are equal to 1. Other elements of the matrix are equal to zero in the initial time instant. A mean value and a covariance matrix of a vector $\Delta z(t)$ are calculated by moment's equations [Malyshev at all, 1996]:

$$\left.\begin{aligned} \dot{m}_{\Delta z} &= A(t)\, m_{\Delta z}, \\ \dot{K}_{\Delta z} &= A(t) K_{\Delta z} + K_{\Delta z} A^T(t) \end{aligned}\right\}. \tag{20}$$

Calculation results obtained using moment's equations for 4-stage launch vehicle were proved using the Monte Carlo simulation technique in application to the Eq. (8). The covariance matrices obtained by both methods are coincided with an acceptable for practical purpose accuracy. But computation time the Monte Carlo simulation is several time greater then the one for the moments equation solution due to the necessity of hundreds launcher motion trajectories simulation.

8. PROBABILISTIC ANALYSIS

Let us consider the probabilistic analysis problem in application to current position and velocity of a vehicle.

Firstly, relationship between the correlation matrix $K_{\Delta z}(t)$ and the criterion functions $\hat{O}_1(t)$ and $\hat{O}_2(t)$ has to be determined.

The function $\hat{O}_1(t)$ can be represented as $\hat{O}_1(t) = \hat{O}_1^1(\xi^3, t)$, where ξ^3 - is the standard Gaussian vector with elements $\xi^3{}_i \in N(0,1), i = \overline{1,3}$.

The function $\hat{O}_2(t)$ can be represented similarly as $\hat{O}_2(t) = \hat{O}_2{}^1(\xi^3, t)$.

Then the probability analysis problem in respect to a disturbed motion trajectory consists in estimation of the probabilities:

$$P_{\varphi_i}(t) = \mathrm{P}\left\{\xi^3 : \hat{O}^1{}_i(\xi^3 t) \le \varphi_i\right\}, i = \overline{1,2}, \tag{21}$$

$$\hat{O}_{\alpha_i}(t) = \min\left\{\varphi_i : P_{\varphi_i}(t) \ge \alpha\right\}, i = \overline{1,2}. \tag{22}$$

A direct solution of these problems requires significant computational time. The *confidential approach* can be suggested for the probabilistic problem solution to reduce of these expenditures.

The problem Eq. (22) is replaced by the equivalent optimization problem (Malyshev, Kibzun, 1987):

$$\hat{O}_{\alpha i}(t) = \min_{E \in \mathrm{E}_\alpha} \max_{\xi^3 \in E} \hat{O}_i(\xi^3, t), i = \overline{1,2}, \tag{23}$$

and by the problem

$$\hat{O}_{\alpha i}(t) = \hat{O}_i^{1-\alpha}(t) = \max_{D \in \mathrm{E}^{1-\alpha}} \min_{\xi \in D} \hat{O}_i^1(\xi^3, t), i = \overline{1,2}. \tag{24}$$

The problem Eq. (21) is replaced by the equivalent optimization problems:

$$P_{\varphi_i}(t) = \max_{\mathrm{E}:\psi(E,t)\le\varphi} \mathrm{P}(E),\ \ \psi(E,t) = \max_{\xi^3\in E} \hat{O}_i^1(\xi^3, t),\ i = \overline{1,2}, \tag{25}$$

$$P_{\varphi_i}(t) = \min_{D:\chi(D,t)>\varphi} \mathrm{P}(D), \tag{26}$$

The algorithm of the directed integration of probability density (Malyshev, Kibzun, 1987) is used for the quantile $\hat{O}_{\alpha_i}(t)$ calculation. The algorithm of a directed integration of a probability density also is used for calculation of a probability $P_{\varphi_i}(t)$.

9. ANALYSIS OF SATELLITE INJECTION ACCURACY

In the probabilistic analysis problem of satellite injection accuracy into an orbit it is necessary to calculate probability and quantile as functions of covariance matrix $K_{\Delta z}(T)$ at the terminal time moment *T*. The function $\hat{O}_3(T)$ can be represented as $\hat{O}_3(T) = \hat{O}_3^1(\xi^6, T)$, where ξ^6 is a standard Gauss vector with elements $\xi_i \in N(0,1), i = \overline{1,6}$. This function is non-linear function of the random vector ξ^6.

The probabilistic analysis problems can be represented as follows:

$$P_{\varphi_3}(T) = P\left\{\xi^6 : \hat{O}_3{}^1(\xi^6, T) \le \varphi_3\right\}, \tag{27}$$

$$\hat{O}_{\alpha 3}(T) = \min\left\{\varphi_3 : P_{\varphi_3}(T) \ge \alpha\right\}. \tag{28}$$

Solving of these problems requires to significant computational time. Using the confidential approach allows reducing that time substantially.

The problem Eq. (28) is replaced by the equivalent optimization problem:

$$\hat{O}_{\alpha 3}(T) = \min_{E\in\mathrm{E}_\alpha} \max_{\xi^6\in\mathrm{E}} \hat{O}_3^1(\xi^6, T), \tag{29}$$

and by the problem:

$$\hat{O}_{\alpha 3}(T) = \hat{O}_3^{1-\alpha}(T) = \max_{D\in\mathrm{E}^{1-\alpha}} \min_{\xi^6\in D} \hat{O}_3^1(\xi^6, t), \tag{30}$$

and the problem Eq. (27) is replaced by the equivalent optimization problems:

$$P_{\varphi_3}(T) = \max_{E:\psi(\mathrm{E},T)\le\varphi} \mathrm{P}(\mathrm{E}),\ \ \psi(E,T) = \max_{\xi^6\in E} \hat{O}_3^1(\xi^6, T), \tag{31}$$

$$P_{\varphi_3}(T) = \min_{D:\chi(D,T)>\varphi} \mathrm{P}(D),\ \ \chi(D,T) = \min_{\xi^6\in D} \hat{O}_3^1(\xi^6, T). \tag{32}$$

The adaptive Monte-Carlo algorithms are used for the quantile $\hat{O}_{\alpha 3}(T)$ and the probability $P_{\varphi_3}(T)$ evaluation.

10. NUMERICAL EXAMPLE

Some numerical results of a probabilistic analysis problem solution are presented in this section.

Quantile and probability of absolute deviations of both velocities and coordinates calculated using the covariance matrixes for along a nominal trajectory are shown in the Fig. 1.

The following characteristic parts in this plot can be selected:

Part AB - the flight with mid-flight thrust of a first stage. The quantile is increased due to effect of random errors of a motor thrust realization as well as due to random errors in aerodynamic characteristics of a vehicle, a random wind in an atmosphere and an atmosphere density;

Part BC – a burning out of a first stage motors. The quantile is decreased because motors are switched off earlier in a case of the greater mid-flight thrust and are switched off later in a case of the smaller mid-flight thrust;

Part CD – the first stage separation. The quantile is increased; the first stage separation event is connected to actual time of first stage motors switching off as well as because the second stage ignition moment is scheduled depending an actual time of the first stage switching off moment;

Part DE - the flight with mid-flight thrust of a second stage, the quantile is increased due to the reasons similar to the characteristic for the part AB;

Part EF - the second stage motor burning-out, quantile is decreased due to reasons, characteristic to the part BC;

Part FG – the second stage separation, quantile is increased due to reasons similar to ones characteristic for the part CD;

Part GH - the flight with mid-flight thrust of the third stage, quantile is increased due to the motor thrust realization errors;

Part HI - the third stage motor burning-out. The quantile is decreased for the reasons similar to the mentioned above for the part BC;

Part IJ – free flight. The quantile is increased due to random initial condition at start moment of this part of a flight;

Part JK - the flight with a mid-flight thrust of the fourth stage, The quantile is increased due to motor thrust realization errors;

Part KL - the fourth stage solid propellant motor burning out. The quantile is decreased; because of the motor is switched off earlier in a case of the greater mid-flight thrust, the motor switched off later in a case of the smaller mid-flight thrust, resulting in partial compensation of deviations.

The quantile of module of a vehicle position deviation from nominal along a trajectory is shown in Fig. 2. It can be seen that this quantile function has the increasing in character.

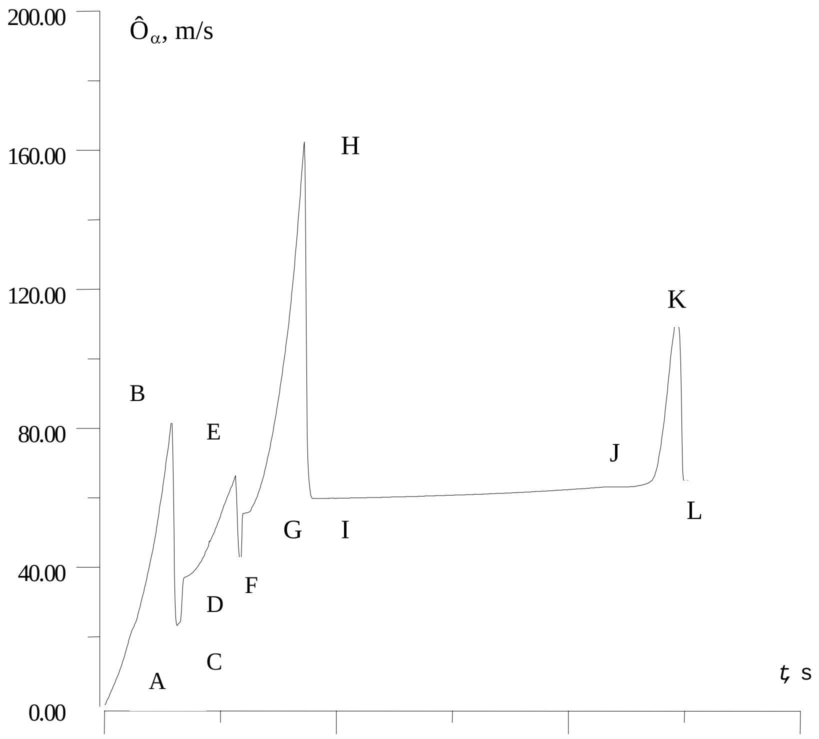

Figure 1 - Quantile of velocities absolute deviation

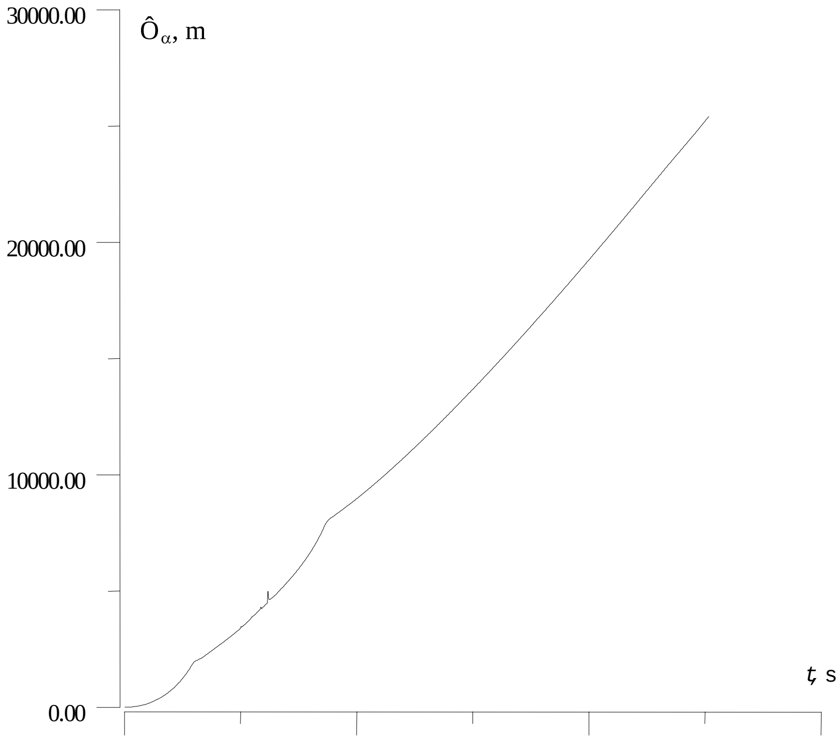

Figure 2 - Quantile of coordinates absolute deviation

The quantile and probability of satellite injection into the orbit calculated utilizing terminal correlation matrixes with use of adaptive algorithm of a Monte-Carlo method are given in Table1.

Table 1. Values of quintiles and probabilities for a satellite insertion errors

Disturbances	Quantile of the order α =0.99	Probability of the level φ =1
Thrust of the first stage motors	0,318	0,9989
Thrust of the fourth stage and spin-up motors	0,432	0,9983
Thrust of the first, second and third stage motors	0,723	0,9979
Thrust of the first and second stage motors	0,516	0,9984
Atmosphere density	0,048	0,9999
Wind	0,051	0,9999
Atmosphere density and wind	0,086	0,9999
Aerodynamic characteristics	0,126	0,9999
Atmosphere density, aerodynamic characteristics and wind	0,195	0,9999
Thrust of the first, second, third and fourth stages and spin-up motors	0,811	0,9975
Total for all disturbances	0,862	0,997

As it can be seen from the Table 1, the probability of satellite injection into the orbit in presence of all disturbances is equal to 0,997, and quantile of a level 0,99 is equal to 0,862.

11. CONCLUSION

The following results are presented in this paper:

1. The new numerical technique of launcher probabilistic analysis is offered. This technique based on confidential approach, joint use of ordinary and statistical linearization method and moment’s equations.
2. Using of the directed integration of probability density algorithm allows decreasing the computational expenditures in 500 times in comparison with standard integration of probability density. The adaptive Monte-Carlo algorithm allows decreasing the computation expenditures in 2,5 times in comparison with the standard Monte-Carlo algorithm.
3. The quantile and probability of absolute deviations of both velocity and position of a vehicle and the quantile and probability of satellite injection into the orbit were calculated.

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