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# INTERPOLATION FUNCTIONS FOR CONVECTION-DIFFUSION PROBLEMS: APPROXIMATIONS OF EXPONENTIAL-BASED FUNCTIONS AND SOLUTON ACCURACY

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***Abstract.*** *This work develops new approximations for the exponential interpolation functions which are used in power-law scheme (PLS) and WUDS. These approximations allow accurate results for near zero grid Péclet numbers and reduce the scheme to upwind interpolation for a Péclet number about 9. The computational cost of these functions is comparable to the usual approximation used in PLS and is less expensive than the common approximations used in WUDS. The solution of one-dimensional and two-dimensional convection-diffusion problems are used to show the good convergence characteristics of the new approximations for the interpolation functions. Questions concerning the usage of nonuniform grids and exponential-based schemes with source term (WUDS-E, PLS-E and LOADS) are also addressed.*

***Keywords:*** *Interpolation, Finite-volumes, Exponential schemes, Accuracy, Grid independence*

## 1. INTRODUCTION

Finite-volume solutions of convection-diffusion problems are still dominated by exponential-type interpolation functions. These schemes are indeed quite accurate for diffusion dominated problems (low Péclet numbers) if the interpolation functions are used without any approximation. However, the related computational cost is too expensive. Thus, approximations are commonly used for the interpolation functions, as the Power-Law Scheme (PLS) (Patankar, 1980) and the Weighted Upstream Differencing Scheme (WUDS) (Raithby and Schneider, 1988). These approximations introduces additional errors which degrades the solution accuracy, specially when one is trying to obtain benchmark solutions for convection-diffusion problems, that is, when a grid independent solution is sought.

It is known that exponential schemes are not well suited for convective-dominated problems (Leonard, 1997), because they revert to the upwind scheme for grid Péclet numbers above 6 (Leonard and Drummond, 1995). Therefore, there is no point in trying to approximate the exponential functions for $P > 6$. Thus, in this work, new functional approximations have been developed which are very good near $P = 0$. This allows good convergence behavior for the numerical schemes if another source of numerical error is absent.

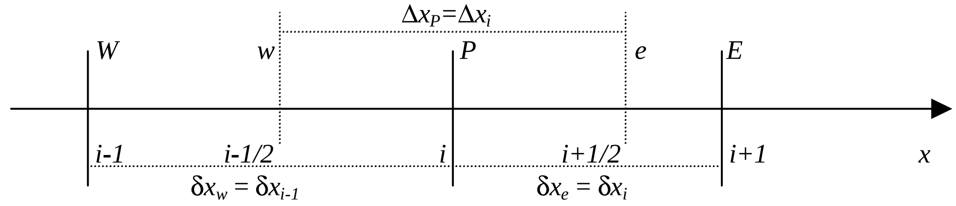



Figure 1 – One-dimensional grid.

## 2. FORMULATION OF EXPONENTIAL SCHEMES

Exponential schemes are based on the local exact solution of the one-dimensional steady convection-diffusion equation. Its conservative form is given by Eq. (1).

$$\frac{\partial}{\partial x}(\rho u \phi) = \frac{\partial}{\partial x}\left(\Gamma_x \frac{\partial \phi}{\partial x}\right) + S \tag{1}$$

where $x$ is the spatial coordinate, $\rho$ is the fluid density, $u$ is the velocity component in $x$ direction, $\Gamma_x$ is the diffusion coefficient for $x$ direction, $\phi$ is the transported variable and $S$ is a source term which is required for some schemes. Equation (1) is integrated over a control volume, giving for the grid shown in Fig. 1, the following flux-conservation equation:

$$\left(\rho u \phi - \Gamma_x \frac{\partial \phi}{\partial x}\right)_e - \left(\rho u \phi - \Gamma_x \frac{\partial \phi}{\partial x}\right)_w = \tilde{S}_P \, \Delta x_P \tag{2}$$

where the tilde indicates volume average. Equation (2) shows clearly the basic task in any finite-volume scheme: the evaluation of convective and diffusive fluxes at cell faces.

The flux estimates are then obtaining by applying Eq. (1) locally between cell nodes, considering $\rho$, $u$, $\Gamma_x$ and $S$ as constants conveniently evaluated at the cell boundary. In dimensionless form, this gives the following boundary-value problem.

$$\begin{cases} P_{i+1/2} \dfrac{\partial \phi}{\partial \xi} - \dfrac{\partial^2 \phi}{\partial \xi^2} = B_{i+1/2} \\ \xi = 0, \phi = \phi_i \text{ and } \xi = 1, \phi = \phi_{i+1} \end{cases} \tag{3}$$

where $B_{i+1/2} = [S/\Gamma_x]_{i+1/2} (\delta x_i)^2$, $\xi = (x - x_i)/\delta x_i$ and $P_{i+1/2} = [(\rho u)/\Gamma_x]_{i+1/2}\, \delta x_i$ is the grid Péclet number. The solution of Eq. (3) for $\phi$ and its derivative are given by

$$\phi(\xi, P_{i+1/2}) = \left[\frac{1}{2} + \alpha(\xi, P_{i+1/2})\right]\phi_i + \left[\frac{1}{2} - \alpha(\xi, P_{i+1/2})\right]\phi_{i+1} + \frac{B_{i+1/2}}{P_{i+1/2}}\left[\alpha(\xi, P_{i+1/2}) + \xi - \frac{1}{2}\right] \tag{4}$$

$$\frac{\partial \phi}{\partial \xi}(\xi, P_{i+1/2}) = \beta(\xi, P_{i+1/2})(\phi_{i+1} - \phi_i) + \frac{B_{i+1/2}}{P_{i+1/2}}\left[1 - \beta(\xi, P_{i+1/2})\right] \tag{5}$$

where

$$\alpha(\xi, P) = \frac{1}{2} - \frac{e^{P\xi} - 1}{e^P - 1} \qquad \text{and} \qquad \beta(\xi, P) = -\frac{\partial \alpha}{\partial \xi} = \frac{P e^{P\xi}}{e^P - 1} \tag{6}$$

If Eqs. (4) and (5) are evaluated at faces $e$ ($\xi_{i+1/2}$) and $w$ ($\xi_{i-1/2}$) the convective and diffusive fluxes of Eq. (2) may be *independently* evaluated. However, it is possible to combine both fluxes in the so-called convective-diffusive flux

$$J(x)=\rho u\phi-\Gamma_x\frac{\partial\phi}{\partial x}\quad\Rightarrow\quad J^*(\xi,P_{i+1/2})=\frac{J\,\delta x_i}{(\Gamma_x)_{i+1/2}}=P_{i+1/2}\phi-\frac{\partial\phi}{\partial\xi}\tag{7}$$

Substitution of Eqs. (4) and (5) into Eq. (7) gives the following expression for the dimensionless flux, $J^*$:

$$J^*(\xi,P_{i+1/2})=\left[A(P_{i+1/2})+P_{i+1/2}\right]\phi_i-A(P_{i+1/2})\phi_{i+1}+\frac{B_{i+1/2}}{P_{i+1/2}}\left[A(P_{i+1/2})+P_{i+1/2}\xi-1\right]\tag{8}$$

where $A$ is defined by

$$A(P)=\beta(\xi,P)-\left[\frac{1}{2}-\alpha(\xi,P)\right]P=\frac{P}{e^P-1}\tag{9}$$

From Eqs. (4), (5) and (8), it is clear that all fluxes depend on cell boundary location ($\xi_{i+1/2}$). For an uniform grid, $\xi_{i+1/2}$ = ½, which is also commonly employed for nonuniform grids. However, this is not strictly correct, because, for a second-order method, node values should be located at cell baricenters in order to correspond to cell-average values.

Equation (8), with $B_{i+1/2}$ = 0, is the exponential scheme given by Patankar (1980), which is truly independent on cell face location. Equations (4) and (5), with $B_{i+1/2}$ = 0, are the basis of the WUDS (*Weighted-Upstream Differencing Scheme*) of Raithby & Torrance (1974). For $B_{i+1/2}$ given as a local approximation of other terms in a multidimensional convection-diffusion problem, Eqs. (4) and (5) are the basis for the WUDS-E (*Weighted-Upstream Differencing Scheme-Extended*) described by Maliska (1995) and for the LOADS (*Locally Analytic Differencing Scheme*) of Wong & Raithby (1979), which differ in the form of evaluating $B_{i+1/2}$. They are also used in the UNIFAES (*Unified Finite Approach Exponential-type Scheme*) of Figueiredo (1997).

## 3. FUNCTIONAL APPROXIMATIONS FOR INTERPOLATION FUNCTIONS

Since the computation of exponential functions are quite expensive, the common practice is to use approximations for $A(P)$ or for $\alpha(½,P)$ and $\beta(½,P)$, assuming that $\xi$ = ½ at cell boundaries regardless of the grid. Thus, the $\xi$ dependence will be dropped from now on.

### 3.1 Classical approximations

Raithby and Schneider (1988) have proposed the following approximations:

$$\alpha_c(P)=\text{sign}(P)\frac{P^2}{10+2P^2}\qquad\text{and}\qquad\beta_c(P)=\frac{1+0.005P^2}{1+0.05P^2}\tag{10}$$

The $\beta_c$ approximation satisfies the $\beta(0)$ and $\beta'(0)$ (first derivative) values but $\beta_c$ for $P\to\infty$ is quite different from $\beta(\infty)$, leading to errors above 20% for $P$ > 5. Although the $\alpha_c$ approximation satisfies $\alpha(\pm\infty)$ and $\alpha(0)$, it does not satisfies $\alpha'(0)$, which implies poor convergence characteristics when one seeks a grid independent solution, because near $P = 0$, the relative approximation error increases without bound.

Patankar (1980) proposed the following approximation for $A(P)$ to be used with Eq. (8), with $B_{i+1/2} = 0$, originating the so-called *power-law scheme* (PLS).

$$A_{PL}(P) = \max\left[0, (1 - 0.1P)^5\right], \quad \text{for } P > 0 \tag{11}$$

For $P < 0$, $A$ is calculated from the property $A(P) = A(|P|)+\max(-P,0)$. Equation (11) is indeed a very good approximation of Eq. (9), satisfying $A(0)$ and $A'(0)$ and truncating $A_{PL}(P)$ to $A(\infty)$ for $P \geq 10$. The use of Eq. (8) with $B_{i+1/2} \neq 0$, calculated as in LOADS, together with Eq. (11) can be defined as the *Power-Law Scheme-Extended* (PLS-E).

However, it is now well accepted that exponential schemes are not adequate for convective-dominated problems (Leonard, 1997), and it is known that when an exponential scheme uses Eqs. (4), (5) and (10) or (8) and (11), it implies the neglect of all physical diffusion for $P$ above about 6, that is, only the numerical diffusion remains in the scheme (Leonard and Drummond, 1995). Although, this is a good approximation for one-dimensional or quasi one-dimensional flows, it implies large amounts of cross-wind artificial diffusion for multidimensional problems.

Therefore, exponential schemes cannot give good results for grid Péclet numbers above about 6, and thus there is no point in trying to well approximate the exponential functions for $P > 6$. On the other hand, the functional approximations must be very good near $P = 0$ when grid-independent results are sought, or the error in approximating the interpolation function will dominated the overall numerical error.

### 3.2 New functional approximations

Good functional approximations around $P = 0$ can be obtained through Taylor series expansions around this point. These series can also be converted to continued fractions, which are sometimes implemented more efficiently. This issue will be seen in the next subsection.

Consider the following Taylor series expansions and their continued fraction representations for $\alpha$, $\beta$ and $A$.

$$\alpha(P) = \frac{1}{8}P - \frac{1}{384}P^3 + \frac{1}{15360}P^5 - \frac{17}{10321920}P^7 + \cdots = \frac{P}{8+}\frac{P^2}{6+}\frac{P^2}{40+}\frac{P^2}{14+}\cdots \tag{12}$$

$$\beta(P) = 1 - \frac{1}{24}P^2 + \frac{7}{5760}P^4 - \frac{31}{967680}P^6 + \cdots = 1 + \frac{P^2}{-24+}\frac{P^2}{(-10/7)+}\frac{P^2}{(2744/11)+}\cdots \tag{13}$$

$$A(P) = 1 - \frac{1}{2}P + \frac{1}{12}P^2 - \frac{1}{720}P^4 + \cdots = 1 + \frac{P}{-2+}\frac{P}{-3+}\frac{P}{2+}\frac{P}{5+}\frac{P}{-2+}\cdots \tag{14}$$

Approximations of several truncation orders have been analyzed for $\alpha$, $\beta$ and $A$. It has been found that $\alpha$ and $\beta$ may be well approximated by low-order series for $P < 6$, while $A$ cannot, for which other functional form must be considered. Modifying Eq. (11) to behave better near $P = 0$, new approximations have been obtained which satisfy $A(0)$, $A'(0)$ and $A''(0)$:

$$A_a(P) = \left[1 - \frac{1}{2a}P - \frac{a-3}{24a^2}P^2\right]^a, \quad \text{for } P > 0 \tag{15}$$

In order to exist a root for $A_a(P) = 0$, which is desirable in order to truncate the approximation to zero for large $P$, the second-order polynomial inside the brackets must have a negative concavity ($a > 3$) and $a$ must be an odd number. For a better implementation efficiency, it is desirable to work with $a = 2^n+1$. This generates successive approximations with $a = 5, 9, 17, 33, \ldots$ , which are very good, especially near $P = 0$.

Table 1 shows the $L_2$-norm and the maximum absolute error for the approximations of $\alpha$, $\beta$ and $A_a$, for $P \in [0,6]$ The errors of $\alpha$ and $\beta$ approximations decrease sharply as the order increases, being much better than those for the $\alpha_c$ and $\beta_c$ approximations. The errors in $A_a$ approximations decrease slowly as $a$ increases. The $A_9$ approximation errors are almost equivalent to those of $A_{PL}$, but with a better behavior near $P = 0$.

### 3.3 Implementation efficiency of the new functional approximations

After analyzing the computational implementation of the $\alpha$ and $\beta$ polynomial approximations, the 7$^{th}$ order $\alpha$ and the 6$^{th}$ order $\beta$ approximations seems to represent the best compromise between accuracy and calculation speed. They can be written as

$$\alpha_7(P) = P\left[\frac{1}{8} + P^2\left(-\frac{1}{384} + \frac{1}{15360}P^2\right)\right] = \frac{P}{8+}\frac{P^2}{6+}\frac{P^2}{40} = \frac{P\left(240+P^2\right)}{1920+48P^2} \tag{16}$$

$$\beta_6(P) = 1 + P^2\left(\frac{-1}{24} + \frac{7}{5760}P^2\right) = 1 + \frac{P^2}{-24+}\frac{P^2}{-(10/7)} = \frac{240-3P^2}{240+7P^2} \cong \frac{1-0.0125P^2}{1+0.0292P^2} \tag{17}$$

Without storing any intermediate result, the $\alpha_7$ value can be calculate through Eq. (16) using 6 products and 2 additions (polynomial formulae) or 6 products, 1 division and 2 additions (rightmost formulae). Equation (17) calculates $\beta_6$ using 5 products and 2 additions (polynomial formulae) or 6 products, 1 division and 2 additions (rightmost formulae). Since a division is calculated much slower than a multiplication or an addition (around 8-10 times slower in most compilers), the polynomial formula is the least expensive for computation. Comparing to $\alpha_c$ and $\beta_c$ computation, Eq. (10), $\alpha_7$ and $\beta_6$ should be calculated even faster than $\alpha_c$ and $\beta_c$. However, Eqs. (16) and (17) can be used only up to $P \approx 9$ without generating a nonphysical result. Thus, they are truncated at $P \approx 9$ by using the fast MAX and MIN FORTRAN functions:

$$\alpha_n = \mathrm{MAX}(-0.5, \mathrm{MIN}(0.5, \alpha_7)) \qquad\qquad \beta_n = \mathrm{MAX}(0, \beta_6) \tag{18}$$

FORTRAN computations with Eqs. (10), (16)-(18) have shown that the evaluation of the $\alpha_n$ and $\beta_n$ using the continued fraction formula of Eqs. (16) and (17) are as fast as $\alpha_c$ and $\beta_c$ calculation, while $\alpha_n$ and $\beta_n$ computation using the polynomial formula are 30-35% faster than $\alpha_c$ and $\beta_c$ evaluation.

Computation of $A_a$ approximations is based on the storage of intermediate values for the value of the second order polynomial inside the brackets of Eq. (15) and its even powers. Even with this procedure, $A(P)$ computation using $A_9$ is 25% slower than the $A(P)$ calculation using the $A_{PL}$ approximation, Eq. (11). The slower computation is compensated by a better convergence behavior of the numerical scheme near $P = 0$. Since $A_9$ becomes less than zero for $P > 11$, the new interpolation function must be calculated through

$$A_n(P) = \mathrm{MAX}(0, A_9) + \mathrm{MAX}(-P, 0) \tag{19}$$

Table 1. Errors in exponential function approximations ($P \in [0,6]$).

| $\alpha(P)$ approximations | | | $\beta(P)$ approximations | | | $A(P)$ approximations | | |
|---|---|---|---|---|---|---|---|---|
| $f(P)$ | $\lVert f-\alpha \rVert_2$ | $\max\lvert f-\alpha \rvert$ | $f(P)$ | $\lVert f-\beta \rVert_2$ | $\max\lvert f-\beta \rvert$ | $f(P)$ | $\lVert f-A \rVert_2$ | $\max\lvert f-A \rvert$ |
| $\alpha_c$ | 0.04462 | 0.04211 | $\beta_c$ | 0.13020 | 0.12196 | $A_{PL}$ | 0.0208 | 0.0147 |
| $\alpha_5$ | 0.02120 | 0.02400 | $\beta_6$ | 0.02594 | 0.03117 | $A_5$ | 0.0432 | 0.0258 |
| $\alpha_7$ | 0.00100 | 0.00137 | $\beta_8$ | 0.00595 | 0.00865 | $A_9$ | 0.0287 | 0.0169 |
| $\alpha_9$ | 0.00003 | 0.00005 | $\beta_{10}$ | 0.00013 | 0.00021 | $A_{17}$ | 0.0195 | 0.0114 |

## 4. NUMERICAL EXAMPLES

In order to check the influence of the interpolation function approximation on the overall numerical error, two problems have been solved using exponential schemes. The first one is the one-dimensional steady convection-diffusion problem given by Eq. (1) with constant coefficients and constant source term. Since the exponential schemes are based on the solution to this problem, the overall error in this case is only due to the approximation of the interpolation function. The second problem is the two-dimensional steady convection-diffusion problem (2D viscous Burgers equation) considering a velocity field which makes a $45^o$-angle with the grid coordinates. This problem with a nonlinear velocity field is commonly used as an example of the numerical error originated when one-dimensional interpolation functions are used. This error usually dominates the overall numerical error.

### 4.1 Steady one-dimensional convection-diffusion problem

Consider Eq. (1) applied to a domain $0 < x < 1$, with $\phi(0) = 0$ and $\phi(1) = 1$, which is discretized in the form of Eq. (2) with constant $S$. Using Eqs. (4) and (5) or (8), the final discretized equation is written as

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + b \tag{20}$$

where, for WUDS-E and PLS-E, $a_P = a_E + a_W$, and

$$a_E = \frac{\Gamma_e}{\delta x_e}\left[\beta_e - \left(\frac{1}{2} - \alpha_e\right)P_e\right] = \frac{\Gamma_e}{\delta x_e}A_e,\; a_W = \frac{\Gamma_w}{\delta x_w}\left[\beta_w + \left(\frac{1}{2} + \alpha_w\right)P_w\right] = \frac{\Gamma_w}{\delta x_w}\left[A_w + P_w\right] \tag{21}$$

$$\begin{aligned} b &= S\left\{\Delta x_P - \delta x_e\left(\alpha_e + \xi_e - \frac{1}{2} - \frac{1-\beta_e}{P_e}\right) + \delta x_w\left(\alpha_w + \xi_w - \frac{1}{2} - \frac{1-\beta_w}{P_w}\right)\right\} \\ &= S\left\{\Delta x_P - \delta x_e\,\frac{A_e + P_e\xi_e - 1}{P_e} + \delta x_w\,\frac{A_w + P_w\xi_w - 1}{P_w}\right\} \end{aligned} \tag{22}$$

Equation (20) can be directly solved by Thomas algorithm (TDMA). WUDS and PLS results are obtained when the terms multiplying $\delta x_e$ and $\delta x_w$ are dropped from Eq. (22). For this problem, WUDS-E and LOADS are equivalent.

Figures 2 and 3 show the results for S = 5, $\Gamma_x = 1$ and $\rho u = 10$ (physical Péclet of 10) using $N$-1 volumes where $N$ = 5, 10, 20, 40, 80 and 160. Uniform and nonuniform grids have been used. The nonuniform grids were obtained through the $x \to x^{1/4}$ transformation of cell faces that concentrates volumes near the x = 1 boundary, where large gradients are present. However, they are, on purpose, extremely concentrated, being somewhat inadequate. This has been done to test the scheme robustnesses. Cell centers were located at volume baricenters.

Figure 2 shows results for $N$ = 5 when the *exact interpolation function*, Eq. (6), has been used. For the uniform grid, very accurate results have been obtained. Furthermore, it shows how important is to consider the actual values for $\xi$ on cell faces in Eq. (22) for WUDS-E in a nonuniform grid, even though they are still approximated by 0.5 in the $\alpha$ and $\beta$ functional approximations. The behaviors of PLS and PLS-E using Eq. (9) are similar to those of WUDS and WUDS-E, respectively.

Figure 3 shows the norm of the solution error for results obtained through WUDS, WUDS-E and PLS-E using the classical approximations ($\alpha_c$, $\beta_c$ and $A_{PL}$) and the proposed approximations ($\alpha_n$, $\beta_n$ and $A_n$). The error norm is defined by

$$\left\|\phi^{N-1} - \phi\right\| = \left[\frac{1}{N-1}\sum_{i=1}^{N-1}\left(\phi_i^{N-1} - \phi_i\right)^2\right]^{1/2} \tag{23}$$

where $\phi^{N\text{-}1}$ is the approximate solution with $N$-1 volumes and $\phi$ is the analytical solution. The uniform grid results show that the usage of ($\alpha_n$, $\beta_n$) leads to a much better convergence behavior for WUDS and WUDS-E than using ($\alpha_c$, $\beta_c$). The usage of $A_n$ instead of $A_{PL}$ in PLS-E shows only a moderate improvement in the convergence behavior. For the nonuniform grid, WUDS accuracy degenerates, regardless of the approximations used for $\alpha$ and $\beta$ (their curves are superposed in Fig. 3b). WUDS-E with ($\alpha_n$, $\beta_n$) behaves worse than for the uniform grid but its convergence is still very good, being much better than those of WUDS-E with ($\alpha_c$, $\beta_c$) on the same grid. The PLS-E results with $A_n$ on the nonuniform grid are also a little worse than those on the uniform grid, and they are comparable to the results obtained with $A_{PL}$ on the same grid. On the other hand, WUDS-E with ($\alpha_c$, $\beta_c$) and PLS-E with $A_{PL}$ behave a little better on the nonuniform grid than on the uniform grid. The WUDS-E with ($\alpha_n$, $\beta_n$) shows the best behavior on both grids.

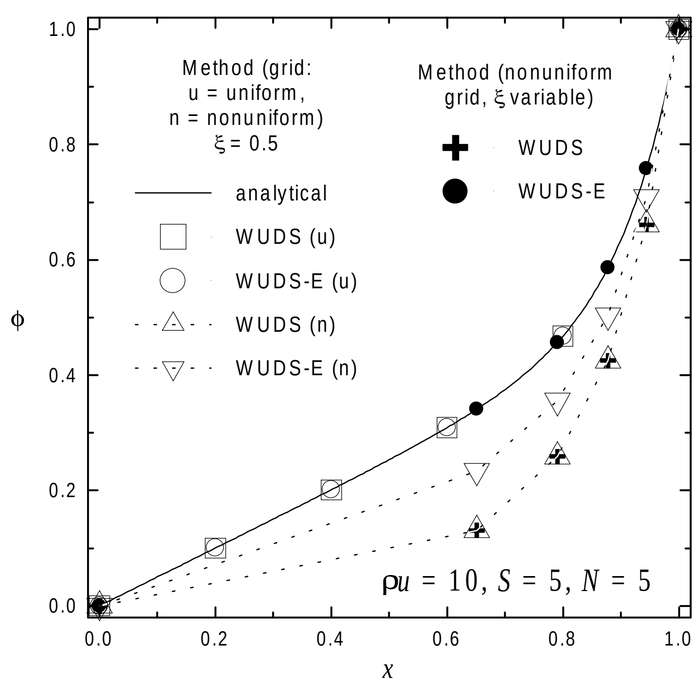
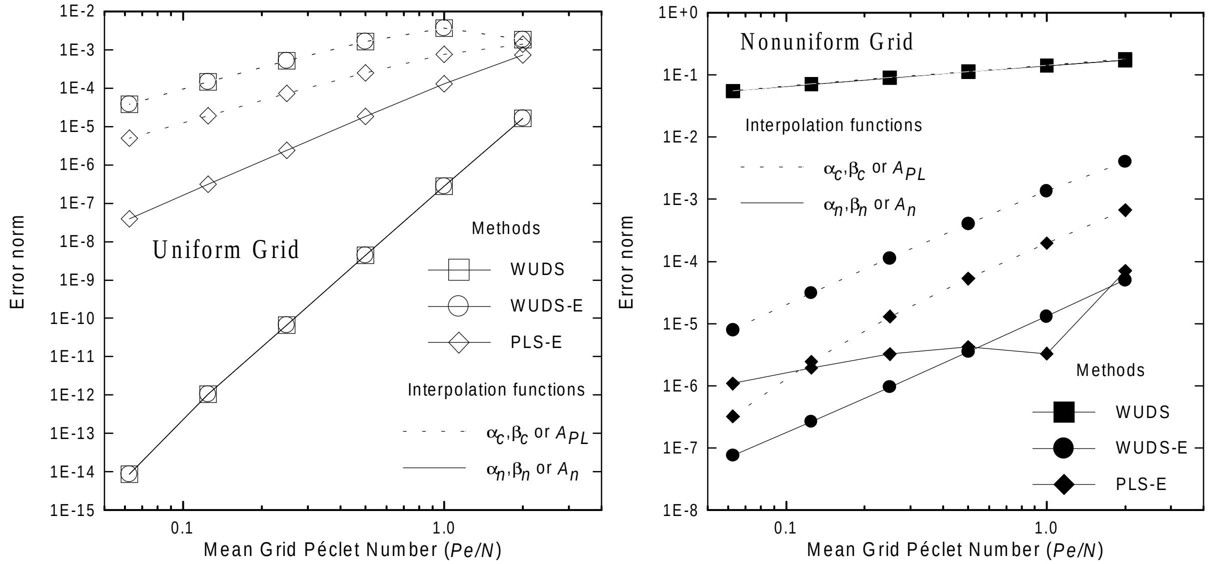



Figure 2 – Steady one-dimensional convection-diffusion problem: grid analysis.




(a) (b)

Figure 3 – Results for the steady one-dimensional convection-diffusion problem: (a) uniform and (b) nonuniform grids.

### 4.2 Two-dimensional viscous Burgers equation

The chosen two-dimensional Burgers problem is given by (Cotta, 1993)

$$u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2},\qquad 0<x<1,\ \ 0<y<1 \tag{24}$$

$$\phi(x,0)=\{1-\exp[u_0(x-1)]\}/\{1-\exp(-u_0)\},\qquad \phi(x,1)=0,\qquad 0\le x\le 1 \tag{25}$$

$$\phi(0,y)=\{1-\exp[u_0(y-1)]\}/\{1-\exp(-u_0)\},\qquad \phi(1,y)=0,\qquad 0\le y\le 1 \tag{26}$$

where $u = v = u_0$ for the linear problem and $u = v = u_0\phi$ for the nonlinear case. For the linear case, the analytical solution is given by (Cotta, 1993)

$$\phi(x,y)=\frac{1-\exp[u_0(x-1)]}{1-\exp(-u_0)}\frac{1-\exp[u_0(y-1)]}{1-\exp(-u_0)} \tag{27}$$

Due to the one-dimensional results presented above, Equation (24) has been discretized using only the methods that utilize the $(\alpha,\beta)$ formulation, that is, WUDS, WUDS-E and LOADS. The final equation can be written as

$$a_P\phi_P=a_E\phi_E+a_W\phi_W+a_N\phi_N+a_S\phi_S+b \tag{28}$$

where $a_P=a_E+a_W+a_N+a_S$, and

$$a_E=\{\Delta y_P[\beta_e-(0.5-\alpha_e)P_e]\}/\delta x_e\ ,\quad a_W=\{\Delta y_P[\beta_w+(0.5+\alpha_w)P_w]\}/\delta x_w \tag{29}$$

$$a_N=\{\Delta x_P[\beta_n-(0.5-\alpha_n)P_n]\}/\delta y_n\ ,\quad a_S=\{\Delta x_P[\beta_s+(0.5+\alpha_s)P_s]\}/\delta y_s \tag{30}$$

$$b = \left\{ \frac{\Delta y_P}{\delta x} B \left[ \frac{1-\beta}{P} - \left( \alpha + \xi - \frac{1}{2} \right) \right] \right\}_w^e + \left\{ \frac{\Delta x_P}{\delta y} B \left[ \frac{1-\beta}{P} - \left( \alpha + \xi - \frac{1}{2} \right) \right] \right\}_s^n \tag{31}$$

where $\{f\}_w^e = f_e - f_w$ and $B$ at each face has been determined from convective-diffusive flux evaluations for LOADS and from approximations for $\phi$ and its derivatives for WUDS-E. For WUDS, $b$ = 0. In $b$ evaluation, Eq. (31), the actual face position, $\xi$, has been used. Equation (28) has been solved by OMSIP (Optimized Modified Strong Implicit Procedure) (Lage 1996a, 1996b), using very strict tolerance criteria. Solutions of linear cases using the *exact interpolation functions* have given the *exact results* to machine accuracy for WUDS and LOADS. The results for WUDS-E are worse, although quite accurate, due to the finite-difference approximations used to evaluate $\phi$ and its derivatives.

Figure 4 shows the error norm results, Eq. (23), for the linear case with $u_0$ = 50 for the uniform grid and a nonuniform grid, obtained by applying the $x \to x^{1/2}$ transformation to volume faces, and for WUDS and LOADS. The usage of ($\alpha_c$, $\beta_c$) leads to poor convergence behaviors, while the ($\alpha_n$, $\beta_n$) results show fast convergence to the exact solution on both grids and for both methods.

The nonlinear case with $u_0$ = 10 has also been solved by WUDS and LOADS in an uniform grid ($N$ up to 80). Both results with ($\alpha_n$, $\beta_n$) are identical (to 4 figures) to those obtained with the *exact interpolation functions*. However, comparing them to the numerical converged solution given by Cotta (1993), it has been found that convergence has not been achieved because the one-dimensional flux approximation dominates the overall numerical error. There is almost no difference between ($\alpha_n$, $\beta_n$) and ($\alpha_c$, $\beta_c$) results for large $N$. In this case the only reason to used ($\alpha_n$, $\beta_n$) over ($\alpha_c$, $\beta_c$) is that their calculation is about 30% faster.

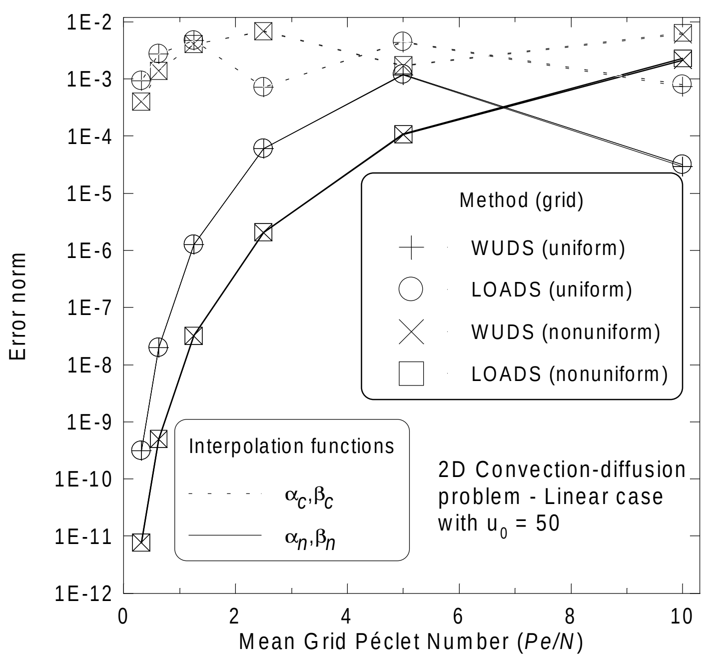



Figure 4 – Results for linear two-dimensional Burgers equation.

## 5. CONCLUSIONS

New approximations for the interpolation functions of exponential schemes (power-law WUDS, LOADS, etc.) have been developed. These approximations are more accurate than previous ones near zero Péclet number, leading to better convergence behavior and allowing grid independent solutions. Their truncation, which leads to the upwind scheme, occurs at a Péclet number about 9. The computational cost of the new $(\alpha,\beta)$ approximations are about 30-35% lower of the classical ones (Raithby and Schneider, 1988), while the cost of the new *A* approximation is about 25% higher than the cost of Patankar's (1980) power-law.

The solutions of linear one- and two-dimensional convection-diffusion problems have shown the better convergence behavior achieved by using the new $(\alpha,\beta)$ approximations. For nonuniform grids and schemes with source term (WUDS-E, PLS-E and LOADS), it has also been shown how important is to consider the actual face position when calculating the source term. Among the tested methods, LOADS has shown the better convergence behavior in both uniform and nonuniform grids. Solutions for the nonlinear two-dimensional Burgers equation have proved that the overall error is dominated by the one-dimensional flux approximation error, caused by the lack of cross-wind terms in these approximations (Leonard et al., 1996). In this case, the error of the interpolation function approximation is not critical.

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**REFERENCES**

Cotta, R. M., 1993, Integral Transforms in Computational Heat and Fluid Flow, CRC Press, Boca Raton, pp. 270-278.

Figueiredo, J. R., 1997, A unified finite-volume finite-differencing exponential-type scheme for convective-diffusive fluid transport equations, RBCM – Journal of the Brazilian Society of Mechanical Sciences, vol. 19, no. 3, pp. 371-391.

Lage, P. L. C., 1996, The Modified Strong Implicit Procedure with Adaptive Optimization of its Iteration Parameter, Numerical Heat Transfer, Part B, vol. 30, n. 3, pp. 255-270.

Lage, P. L. C., 1996, Application of the Optimized Modified Strong Implicit Procedure to Nonlinear Problems, Numerical Heat Transfer, Part B, vol. 30, n. 4, pp. 423-435.

Leonard, B. P., 1997, Bounded Higher-Order Upwind Multidimensional Finite-Volume Convection-Diffusion Algorithms, in Advances in Numerical Heat Transfer, eds. Minkowycz, W. J., & Sparrow, E. M., Taylor & Francis, New York, vol. 1, Chap. 1.

Leonard, B. P., & Drummond, J. E., 1995, Why you should not use 'Hybrid' 'Power-law' or Related Exponential Schemes for Convective Modelling-There are Much Better Alternatives, International Journal for Numerical Methods in Fluids, vol. 20, pp. 421-442.

Leonard, B. P., Lock, A. P., & MacVean, M. K., 1996, Conservative Explicit Unrestricted-Time-Step Multidimensional Constancy-Preserving Advection Schemes, Monthly Weather Review, vol. 124, pp. 2588-2606.

Maliska, C. R., 1995, Transferência de Calor e Mecânica dos Fluidos Computacional, LTC, Rio de Janeiro.

Patankar, S. V., 1980, Numerical Heat Transfer and Fluid Flow, Hemishere, New York.

Raithby, G. D., & Torrance, K. E., 1974, Weighted-upstream differencing schemes and their application to elliptic problems involving fluid flow, Computer & Fluids, vol. 2, pp. 191-206.

Raithby, G. D., & Schneider, G. E., 1988, Elliptic Systems: Finite-Difference Method II, in Handbook of Numerical Heat Transfer, eds. Minkowycz, W. J., et al., John Wiley, New York, Chap. 7, pp. 241-291.

Wong, H. H., & Raithby, G. D., 1979, Improved finite-difference methods based on a critical evaluation of the approximation errors, Numerical Heat Transfer, vol. 2, pp. 139-163.