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IMPLEMENTATION OF A FINITE ELEMENT RELIABILITY METHOD

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Abstract. *This paper presents a method for evaluating probabilities of failure (P_f) of mechanical components and structures using finite element models. Design variables are characterized as random variables. A limit state equation is written for each significant failure mode in terms of loading effects. The P_f is calculated by integrating the joint probability density function over the failure domain. This solution is approximated by means of the first order reliability method. In the case of multiple failure modes, a bi-modal approximation or Monte Carlo simulation is employed. Simulation is performed using response surfaces and an importance sampling scheme, with one sampling function for each design point. The method was programmed in form of a set of subroutines for the ANSYS analysis software. The method has shown to be robust and efficient. The reliability calculations are automated and require only two additional task in comparison to a deterministic analysis: construction of a parametric FE model and statistical description of random variables. The solutions for two problems from literature and for an application to an eye-bar cable suspension bridge are presented.*

Key-words: *Structural Reliability, Finite Element method, Stochastic Finite Element Method, FORM, SORM, Monte Carlo Simulation.*

1. INTRODUCTION

The solution of engineering problems involves basically four types of uncertainties: physical, statistical, modeling and phenomenological uncertainties. Those uncertainties are due to: natural variability of some design variables, uncertainty in data estimation, uncertainty introduced by modeling simplifications and uncertainty with respect to the problem being solved. Worried with the effects of uncertainty in the design of structures

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and mechanical components, engineers have developed Structural Reliability methods. This methods take uncertainty into account and allow the calculation of a quantified measure of the structures safety: the failure probability (P_f).

On the other hand, finite element (FE) methods have made a revolution in structural analysis. The precision, efficiency, accuracy of these methods cannot be neglected. There is a natural tendency, therefore, to incorporate FE analysis tools into structural reliability methods, bridging the gap between these methods.

The first progress in this direction was the development of the so-called Second Moment Stochastic Finite Element (SFE) methods (Liu and Kiureghian, 1997; Beck, 1999). In these methods, the FE equilibrium equations are expanded in series and first order terms are considered. This expansion results in expressions for the mean and for the standard deviation of the structural response. Because of simplifications made in deriving the solutions, these methods are limited to problems with normal variables, with small standard deviations and with small correlation coefficients.

In recent years, a method known as Finite Element Reliability Method (FERM) has been developed. This method does not present the limitations of previous methods. It employs standard structural reliability methods as FORM, SORM and Monte Carlo simulations to calculate failure probabilities, as will be explained in the sequence. Moreover, it can be implemented using commercial FE software's, as was done in this paper.

2. FRAMEWORK FOR THE FERM METHOD

The basic problem of structural reliability is the evaluation of the failure probability. A limit state function is defined in terms of some loading effects **s(z)**, which are function of the random variable vector **z** and of some threshold variables like yielding stress, ultimate stress or maximum displacement:

$$g(\mathbf{s}(\mathbf{z}),\mathbf{z}) = \mathbf{z}_t - \mathbf{s}(\mathbf{z}) = 0 \quad (1)$$

The limit failure equation divides the space of design variables in a safe and a failure domain. The failure probability is evaluated by integrating the joint probability distribution function (PDF) of the random variables over the failure domain:

$$P_f = \int_{g(\mathbf{z})\leq 0} f_{\mathbf{Z}}(\mathbf{x})d\mathbf{x} \quad (2)$$

When using a FE model to solve the mechanical problem, the analytical expression of $g(\mathbf{s}(\mathbf{z}),\mathbf{z})$ is not known. Because the joint PDF and the analytical expression of the limit state equation are not known, approximate methods like FORM, SORM, and Monte Carlo Simulation are used to solve equation (2), as will be shown later.

One great advantage of the FERM method compared to the second moment SFE is the possibility to use the FE model of the problem as a black box, as shown in figure 1. This is possible because the mechanical and probabilistic solutions are independent of each other. This feature allows the use of commercial FE software to solve the mechanical part of the problem.

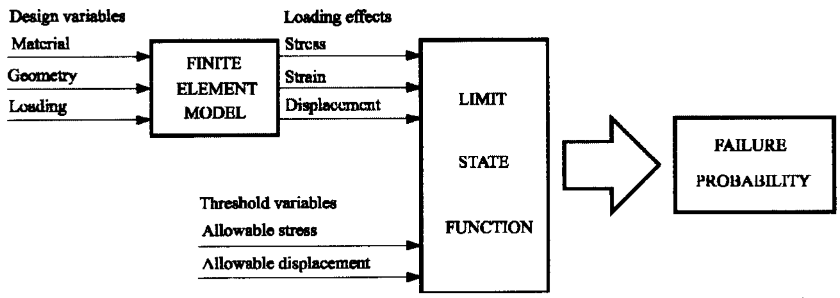

Figure 1: Use of the FE model as a black box in the FERM method.

2.1 The first order reliability method

The first order approximation to equation (2) is obtained by replacing the original limit state function by a tangent hyper-plane at the most probable failure point, the design point. The approximation is made in a space of standard uncorrelated normal variables. Because of the rotational symmetry of the joint PDF in this space, the design point is the point over the limit state function closest to the origin. The minimal distance between the design point and the origin of this space is the reliability index β. The first order estimate of equation (2) is given in terms of β:

$$P_{f1} = \Phi(-\beta) \tag{3}$$

where Φ is the cumulative standard normal distribution. Because of the minimum property of the reliability index, it can be found through an appropriate optimization algorithm. The optimization problem in FORM is stated as:

$$\begin{aligned} &\text{minimize}: \quad \beta = \sqrt{\mathbf{y}^t \cdot \mathbf{y}} \\ &\text{subject to}: \ g(\mathbf{y}) = 0 \end{aligned} \tag{4}$$

In this paper, the modified HLRF algorithm is used. A search direction is obtained by a single expression:

$$\mathbf{d}_k = \frac{\mathbf{y}_k \cdot \nabla g(\mathbf{y}_k) - g(\mathbf{y}_k)}{\nabla g(\mathbf{y}_k)^t \cdot \nabla g(\mathbf{y}_k)} \cdot \nabla g(\mathbf{y}_k) - \mathbf{y}_k \tag{5}$$

where $\nabla g(\mathbf{y}_k)$ is the gradient of the limit state function with respect to random variables, calculated in the standard normal space. A linear search in $\mathbf{d}_k$ is done until a sufficient decrease in a merit function is obtained. A good merit function, that employs only available information, is:

$$m(\mathbf{y}_k) = \frac{1}{2}\left|\mathbf{y}_k - \frac{\mathbf{y}_k \cdot \nabla g(\mathbf{y}_{k-1})}{|\nabla g(\mathbf{y}_{k-1})|^2} \cdot \nabla g(\mathbf{y}_{k-1})\right|^2 + \frac{1}{2} c g(\mathbf{y}_k)^2 \tag{6}$$

where c is a constant.

The problems original joint PDF has to be mapped into the standard normal space. This is done through a probabilistic transformation:

$$\mathbf{y}(\mathbf{z}) = \mathbf{J}_{y,z} \cdot \{\mathbf{z} - \mathbf{M}^{neq}\} \tag{7}$$

where $\mathbf{J}_{y,z}$ is the Jacobian of the transformation:

$$\mathbf{J}_{y,z} = \mathbf{L}^{-1} \cdot (\mathbf{D}^{neq})^{-1} \tag{8}$$

and:

$\mathbf{M}^{neq}$ is the vector of means of equivalent normal distributions;

$\mathbf{D}^{neq}$ is the diagonal matrix whose elements are the equivalent standard deviations;

$\mathbf{L}^{-1}$ is obtained through a Choleski decomposition of the correlation matrix.

The probabilistic transformation is composed of tree parts. One is the calculation of equivalent marginal normal distributions, which eliminates asymmetry present in other PDF's. The second is an ortogonalization that eliminates correlation between random variables through the Choleski decomposition. The last transformation is Nataf's model (Kiureghian et all, 1986), used to combine the two previous ones.

2.2 Linearized bi-modal failure bounds

Structural components or structures can usually fail due to multiple failure modes. In this case, one limit state function has to be constructed for each failure mode. The FORM method can be used to evaluate single failure probabilities. These probabilities are used to calculate the failure probability of the component, following the rules for a series system.

The joint failure probabilities (simultaneous failure in two failure modes) cannot be evaluated directly. They are estimated based on the following procedure. Each limit state function is linearized at its design point. Gradients of the limit state functions are used to obtain approximate correlation coefficients between two failure modes. The joint failure probabilities are estimated in terms of the probabilities of some events A and B, which can be determined from the correlation coefficients.

Because of the use of events A and B to approximate the joint failure probability, only bounds to the failure probability can be obtained. In some cases (large individual P_f's, highly correlated failure modes), this bounds can be quite large. Therefore, Monte Carlo simulation methods should be used when multiple failure modes are present.

2.3 Response surface method

When Monte Carlo simulation is used and the number of samples is large, the solution of the detailed FE model of the problem for each sampled point gets expensive. In these cases, a simplified model of the mechanical problem may be used.

In response surface methods, the detailed numerical FE model of the problem is replaced by an approximate polynomial model. The polynomial model is obtained by fitting a general surface to the original limit state function at the design point. A second order surface with cross terms is obtained by:

$$g_{rsm} \approx a + \mathbf{z} \cdot \mathbf{b} + \mathbf{z}^t \cdot \mathbf{c} \cdot \mathbf{z} \tag{9}$$

where a, **b** and **c** contain the coefficients to be determined. In this work, the **b** vector and **c** matrix are approximated by the gradient and the Hessian of the limit state function, which are calculated through central finite difference schemes.

2.4 Monte Carlo simulation

Simple Sampling: In simple Monte Carlo simulation, samples of the vector **z** are generated according to the original PDF $f_{\mathbf{z}}(\mathbf{z})$. The state of the structure (either failure or survival) for each sample point is investigated. An indicator function is used, where:

$$\begin{aligned} I[g(\mathbf{z}_i)] &= 1 \;\text{ if }\; \mathrm{g}(\mathbf{z}) \leq 0 \\ I[g(\mathbf{z}_i)] &= 0 \;\text{ if }\; \mathrm{g}(\mathbf{z}) > 0 \end{aligned} \tag{10}$$

The failure probability is estimated by counting the number of samples generated in the failure domain:

$$\overline{P}_f = \frac{\sum_{i=1}^{n_{si}} I[g(\mathbf{z}_i)]}{n_{si}} \tag{11}$$

Because of the statistical sampling error, simulation results are subjected to a variance, given by:

$$Var(\overline{P}_f) = \frac{\sum_{i=1}^{n_{si}} (I[g(\mathbf{z}_i)] - \overline{P}_f)^2}{n_{si} \cdot (n_{si} - 1)} \tag{12}$$

Due to the small P_f of real world structures, a large number of samples is required, in order to keep the variance within specified limits. The variance can be reduced by increasing the number of samples or by using importance sampling techniques, which shift the sampled points to the failure domain.

Importance Sampling: When using importance sampling, a sampling function $h_{\mathbf{z}}(\mathbf{z})$ is used to generate the samples. Each sampled point is associated to a sampling weight $w_i = f_{\mathbf{z}}(\mathbf{z}_i) / h_{\mathbf{z}}(\mathbf{z}_i)$, and the failure probability is given by:

$$\overline{P}_f = \frac{1}{n_{si}} \cdot \sum_{i=1}^{n_{si}} I[g(\mathbf{z}_i)] \cdot \frac{f_{\mathbf{z}}(\mathbf{z}_i)}{h_{\mathbf{z}}(\mathbf{z}_i)} \tag{13}$$

Importance sampling using design points is a very efficient sampling scheme. This technique makes intelligent use of a priori knowledge: the location of the most important failure regions, around the design points. The sampling function for multiple failure modes is constructed following Schüeller and Stix (1987). The design points are obtained by solving the optimization problem, as in equations (4) and (5). The FORM estimate of the P_f's are used as a clue to the relative importance of each design point. Simulation weights p_i are calculated based on FORM results:

$$p_i = \frac{\Phi(-\beta_i)}{\sum_{i=1}^{k} \Phi(-\beta_i)} \tag{14}$$

The sampling function has one bulge over each design point and is flat elsewhere, as shown in figure 2.

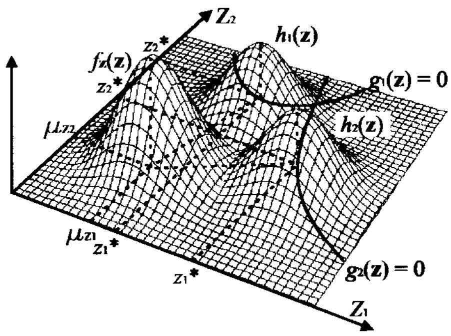

Figure 2: Importance sampling using design points.

Each bulge is obtained by shifting the original PDF to the respective design point. The bulges are scaled according to the weights p_i, and the sampling function results:

$$h_Z(\mathbf{z}) = \sum_{i=}^{k} p_i \cdot h_{iZ}(\mathbf{z}) \tag{15}$$

When importance sampling is used, the required number of simulations is almost independent on the order of magnitude of the P_f. The importance sampling scheme is particularly interesting when used in context of the FERM method, when simulation is performed using response surfaces. With this sampling scheme, the simulation is mostly performed around the design points, where the response surfaces where fitted. For multiple failure modes this technique is even more promising, since the simulation is performed only once for all limit state function. The P_f's with respect to each individual failure mode and with respect to component failure are easily evaluated. The simulation using a second order response surface can significantly improve the linearized bi-modal bounds.

3.0 RESULTS

The FERM method presented in previous section was implemented in Fortran routines, which are compiled and linked to the ANSYS finite element program. The design optimization module of ANSYS is used for this purpose (ANSYS, 1992). The resulting program was called ACE-Pro – a practical tool for evaluating failure probabilities of real world structures (Beck and da Rosa, 1999). The reliability analysis using ACE-Pro requires only two additional tasks comparing to a deterministic FE analysis: the construction of a parametric FE model and the statistical description of random variables. In the parametric model, each random variable is set as a parameter, whose value can change during the analysis.

Several examples from literature were solved using ANSYS and the ACE-Pro module, in order to analyze the performance of the proposed FERM method. The solution for two examples is presented. An application example of the reliability analysis of an eye-bar cable suspension bridge is also presented.

3.1 13 bar truss example

This example was taken from Siddall (1972). 13 limit state functions are considered, one for the yielding of each bar. The yielding stress of each bar is considered to be an independent random variable, with log-normal distributions. The loads P are also considered to be one random variable.

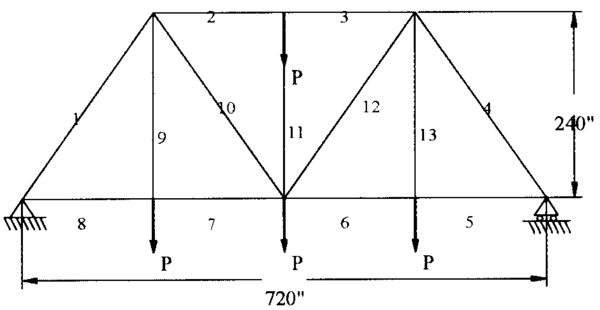

Figure 3: 13 bar truss.

The problem was solved analytically by Siddall using load factors, which relate the load on each bar with the applied loads P. This solution was only possible because the yielding stresses were considered to be uncorrelated. The data for this example can be obtained in Beck (1999). The results are summarized in table 1.

Table 1(a): Results for the 13 bar truss example (series system analysis).

Method	P_f	δ / tol.	n_{si} / n_{iter}
FORM – BMF	$0.1463 < P_f < 0.4154$	10^{-4}	5 per g(**z**)
MCS -SI	0.2434	0.008	5 10^4
MCS - AI	0.2447	0.004	5 10^4

Table 1(b): Results for individual bars.

Bars	P_f (FORM)
1 and 4	3.3 10^{-4}
2 and 3	3.5 10^{-7}
5 to 8	0.1463
9, 11 and 13	7.3 10^{-7}
10 and 12	3.0 10^{-3}

Results show that the failure of the truss is mainly governed by failure of bars 5 to 8. The results for the series system analysis agree with the results obtained by Siddall (P_f = 0.2435). The linearized bi-modal limits (BMF) are very large because of the existence of four equally likely failure modes and because of the large failure probability of these modes.

The safety factor for bar 5 is 1.12. In this example the relation between the safety factor and the failure probability is shown. The problem was solved repeatedly varying the mean value of the random variables. Failure probabilities of bar 5 and of the truss are compared with the safety factor of bar 5 in figure 4. The figure shows a highly non-linear behavior, plotted in both linear logarithmic scales. The figure shows how dangerous it can be to design structures relying only on safety factors.

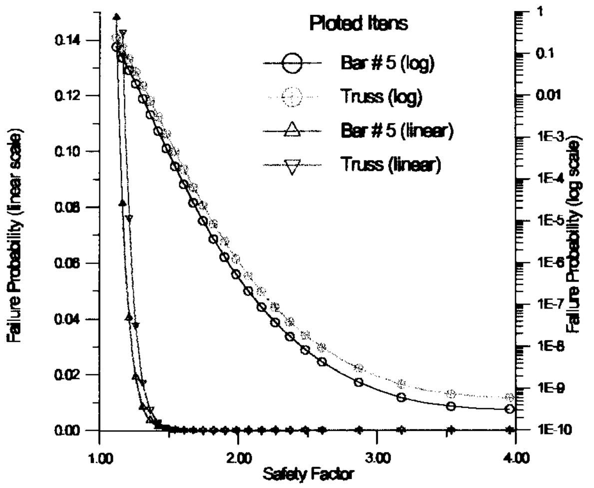

Figure 4: Relation between safety factor and failure probability.

3.2 Built-up column example

This is a very realistic example taken from Liu and Kiureghian (1989). The example involves 22 random variables: two random elasticity modules, for the struts and for the braces and battens, two random forces F1 and F2 (according to figure 5) and 18 nodal coordinates. The vertical coordinate of each node is deterministic, but the horizontal one is random. These random variables take into account imperfections in the construction of the column.

The failure probability due to buckling of the column is considered. The limit sate equation is written in terms of the horizontal displacement at the midspan of the column (node 10). This solution requires a non-linear FE analysis because of the large deflections of the column. Results are presented in table 2.

Table 2: Results for the built-up column example.

Reference	Method	P_f	Tolerance	n_{iter}
Liu and Kiureghian	FORM	5.0 10^{-3}	?	8
ACE-Pro	FORM	1.1 10^{-4}	10^{-3}	8

The results agree with each other, considering that the non-linear solution model used in both works were not the same. Liu and Kiureghian included non-linear effects in terms of material non-linearity's. In the present analysis, geometrical non-linearity due to large deflections was considered in ANSYS.

This example showed one drawback of the finite difference scheme used for the gradient and Hessian computations. Because the number of random variables is large and the calculation of each gradient component requires a non-linear FE analysis, the computation time boomed. Nevertheless, the computation time for this particular example was still reasonable. The FORM estimate was obtained in two minutes and the response surface was constructed in 18 minutes. Convergence problems were encountered for threshold displacements larger than 10 inches.

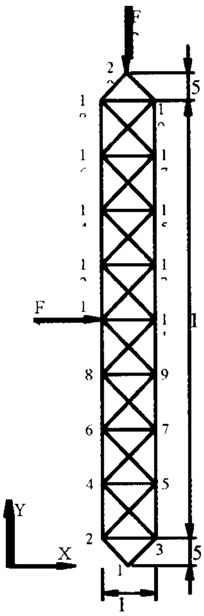

Figure 5: Built-up column.

4. APPLICATION EXAMPLE – THE HERCÍLIO LUZ BRIDGE

The ACE-Pro module was employed in a simplified reliability analysis of the eye-bar cable suspension chain of the Hercílio Luz bridge. This bridge, located in the city of Florianópolis, SC, Brazil, was built in 1926. The bridges suspension chain is composed of sets of four parallel eye bars, connected by pins. A volume plot of part of the bridges 3D beam model used in the analysis is shown in figure 6. The model has 2275 elements and 8866 degrees of freedom. Non-linear stress stiffening effects where considered in this analysis.

For the simplified reliability analysis, only five random variables where considered: the distributed live load (q), the yielding stress (S_y), the ultimate stress (S_u), the actual level of corrosion (area reduction) of the bars (γ) and the dead weight. The uncertainty in dead weight was modeled as an uncertainty in the gravity acceleration constant g. Table 3 resumes data for random variables.

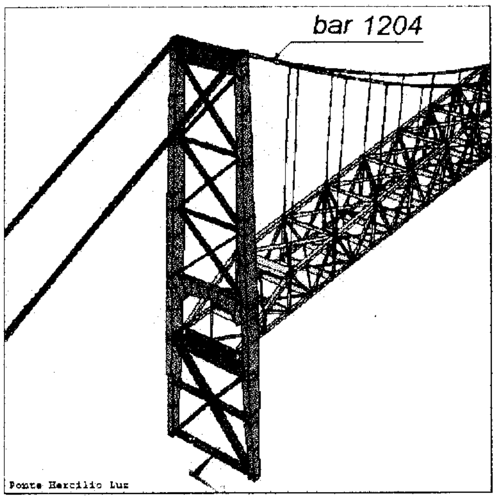

Figure 6: Volume plot of the bridges 3D beam model.

Table 3: Data for the bridge analysis.

Variable	Distribution	Mean	Dispersion	Unit
q	Normal	$1.4\ 10^5$	0.140	N
S_y	Weibull (minimum)	582.60	0.06	MPa
S_u	Weibull (minimum)	848.00	0.05	MPa
γ	Weibull (maximum)	0.10	0.138	-
g	Normal	9.81	0.100	m/s^2

Several situations where analyzed in this study. Because of size constrains, only the results for yielding and rupture of bar 1204 (as shown in figure 6) are presented. Complete results of the analysis can be obtained in (Beck, 1999). Results are shown in table 4.

Table 4: Results for the bridge analysis.

Analysis case	Method	P_f
Yielding	FORM	$6.6\ 10^{-6}$
Rupture	FORM	$1.7\ 10^{-9}$

The computation time required for the solution was of 20 minutes for each limit state equation. The results show that the bridge is safe against the considered failure modes. Further deepening of the analysis should consider more realistic failure modes. The simulation results agreed with FORM, showing that in spite of the non-linearity of the analysis, the limit state functions are rather linear. The convergence of the solution for this case study shows ACE-Pro's ability to handle large scale real world problems.

5. CONCLUDING REMARKS

In this paper, it was shown how structural reliability methods are incorporating FE analysis tools. The traditional second moment SFE method was briefly explained and some limitations where pointed out. The novel FERM method was presented as an alternative to second moment methods and explained in detail. The FERM method was programmed and implemented as a module for the ANSYS program. Some examples were solved using this module. The FERM method is robust and efficient, since it converged for all examples analyzed, with a low number of iterations. The method is robust because it applies to problems with any type of continuous PDF's, for correlated or uncorrelated variables, and with no limitation on the order of standard deviations.

The booming of the computation time verified in the built-up column example show future improvements that are tantamount. The implementation of direct differentiation methods for gradient calculations can significantly speed the convergence of FERM solutions, specially in non-linear problems. The convergence of the solution for the bridge application example shows ACE-Pro's ability to handle real world engineering problems.

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