COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

A NOTE ABOUT THE MECHANICS OF COLLISION BETWEEN TWO ROUGH SOLID BODIES

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Abstract: *Slipping between two rough solid bodies, moving in any manner, during collision is analyzed considering Coulomb's Law of dry friction and negligible deformation in the region of impact. Only the evolution of the tangential collision forces and velocity is discussed. It is found that a vector treatment of the problem simplifies existing geometrical views of the problem, opening ways to generalizations. Plastic, inelastic and perfectly elastic bodies are considered.*

Keywords: *Collision, Friction*

1. INTRODUCTION

Collision between two rough solid bodies was studied with some details by Routh (1897), Beghin (1952), Levi-Civita & Amaldi (1937) and several other authors, must works dating at the end of the past century and earlies 1900 (e.g. Darboux, 1889; Mayer, 1902; Pérès, 1926).

Routh's approach, as well as Levi-Civita & Amaldi's, follows the evolution of the impulses, based on the dynamics of collision and Newton's definition of the coefficient of restitution, always relying on a geometric interpretation of the path followed by the impulses.

Beghin's treatment actually follows the velocity and acceleration of the point of impact, applying the concept of totally inelastic collision, with terminal velocity of the point of impact being zero, and totally elastic collision, with inversion of the normal component of the velocity of that point.

Not many works have been lately developed on the subject, although in a surprising paper by Derby and Fuller (1999), with an equally surprising title "Reality and Theory in a Collision" the authors "discover" that friction plays an important role in the collision of two billiard balls. Their results could be well obtained by an elaborate theory, available since about 100 years. Actually, the fact that the trajectory of the billiard balls is parabolic until slipping on the supporting table stops, being rectilinear afterward, follows from a simple theoretical calculation (Giacaglia, 1967). The fact that friction is a very important force, especially during collision, was demonstrated by several examples in the early literature, in simple cases by Levi-Civita & Amaldi (op. cit.) and Beghin (op. cit.).

One of the complicating factors in the available literature is the mathematics used in the theoretical development of the physical phenomena involved, where a vector and matrix formalism is completely absent. The basic reasoning of this work is based on Beghin's work (op. cit. ,Chapter XV). The procedure consists in following the evolution of the velocity of the point of contact during the time of collision, by integrating the dynamical equations of each body resulting from the motion of the center of mass, the equations for the angular momentum and the law of friction. It has to be assumed, as usual, that during the collision, the acting forces of contact are very large, that is, their impulse is finite during a very short time interval. In vector and matrix notation, components will be named with subscripts *1, 2* or *3*.

2. EQUATIONS OF MOTION

Let for any of the intervening bodies $\vec{v}_G$ be the velocity of the center of mass, $\vec{v}_O$ the velocity of slipping, *G* the center of mass, *O* the point of contact at collision, $\vec{F}$ the force of contact per unit of mass, satisfying the law of action and reaction at any instant, $\vec{w}$ the vector angular velocity, and J_G the inertia matrix relative to the center of mass, in a coordinate system with origin at *O*, the x_3 axis normal to the surface of contact at *O* and plane Ox_1x_2 tangent to this surface. The equations of motion for any of the two bodies may be written as follows:

$$\begin{aligned} \dot{\vec{v}}_G &= \vec{F} \\ \dot{\vec{\omega}} &= -J\Gamma\ \vec{F} \end{aligned} \tag{2.1}$$

where matrix J is symmetric and inverse of matrix J_G., while Γ is skew-symmetric. The velocities of *O* and *G* are related by Poisson's Law

$$\vec{v}_O = \vec{v}_G + \Gamma\vec{\omega} \tag{2.2}$$

From the above equations one finds

$$\dot{\vec{v}}_O = (I - S)\vec{F} \tag{2.3}$$

where I is the identity matrix and $S = \Gamma\ J\ \Gamma$ is a symmetric matrix. Except for Coulomb's Law of dry friction and Newton's empirical Law for the coefficient of restitution, applicable at the point of impact, Eq. (2.3) is sufficient for the analysis of the problem. Obviously, slipping is present whenever the relative velocity of point *O* of one body with respect to point *O* of the other is not zero. For the equation giving the vector angular velocity, it is assumed that pivoting friction is negligible, that is, the contact area is very small, if not zero. It is noted that Eq. (2.3) may be written as

$$\dot{\vec{v}}_O = \frac{1}{2} grad_{\vec{F}} \Phi \tag{2.4}$$

where

$$\Phi(\vec{F}) = \vec{F}^T\vec{F} - \vec{F}^T S\vec{F} \tag{2.5}$$

The equation

$$\Phi(\vec{r}) = r^2 - \vec{r}^T S\,\vec{r} \ \ = 1 \tag{2.6}$$

represents a surface Σ and $\dot{\vec{v}}_O$ is orthogonal to this surface in view of Eq. (2.4). The line drawn from point *O* and parallel to $\vec{F}$, intercepts surface Σ at point P defined by

$$P - O = \lambda \vec{F} \tag{2.7}$$

where $\lambda = (F^2 - \vec{F}^T S \vec{F})^{-1/2}$. This geometrical interpretation was introduced by Beghin,.

3. MOTION UNDER SLIPPING CONDITIONS

When the slipping velocity is not zero, Coulomb's Law gives the magnitude and direction of the tangential force during impact and as long as slipping occurs.

Let μ be the dry friction coefficient between the colliding bodies. It follows that

$$\vec{F} = [-\mu \, \hat{v}_S + \hat{x}_3 \,] F_3 \tag{3.1}$$

where $\hat{v}_S$ is the unit vector along the slipping velocity $\vec{v}_S$, given by

$$\vec{v}_S = \vec{v}_O - v_{O3} \hat{x}_3 \tag{3.2}$$

From Eq. (1.3) one finds, for $k = 1, 2, 3$

$$\dot{v}_{O\,k} = F_k - \sum_j S_{kj} F_j \tag{3.3}$$

By taking into account Eq. (3.1), it follows that

$$\begin{aligned}
\dot{v}_{O1} &= \{ \mu \, [-(1 - S_{11}) v_{O1} + S_{12} v_{O2}] (v_{O1}^2 + v_{O2}^2)^{-1/2} - S_{13} \} F_3 \\
\dot{v}_{O2} &= \{ \mu \, [-(1 - S_{22}) v_{O2} + S_{12} v_{O1}] (v_{O1}^2 + v_{O2}^2)^{-1/2} - S_{23} \} F_3 \\
\dot{v}_{O3} &= \{ \mu \, [S_{13} v_{O1} + S_{23} v_{O2}] (v_{O1}^2 + v_{O2}^2)^{-1/2} + (1 - S_{33} \} F_3
\end{aligned} \tag{3.4}$$

With obvious definitions, Eqs. (3.4) may be written as

$$\frac{dv_{O1}}{n_1(v_{O1}, v_{O2})} = \frac{dv_{O2}}{n_2(v_{O1}, v_{O2})} = \frac{dv_{O3}}{n_3(v_{O1}, v_{O2})} \tag{3,5}$$

These are the differential equations of a curve defining the trajectory of point *M* defined by

$$M - O = \vec{v}_O \tag{3.6}$$

while vector $\vec{n}$ is normal to the surface defined by Eq. (2.6). It is seen that following the path of point *M*, as given by Eqs. (3.5), one follows the evolution of the velocity of contact point *O* during impact. It is also important to note that during impact, the two colliding bodies remain in contact one with each other, so that the relative velocity along the common normal $\hat{x}_3$ is zero, during the deformation phase as well as during the restitution phase. Again, it is stressed that deformation due to tangent friction forces are neglected. From the initial value of the velocity of point *O*, integration of Eq. (3.5) will give the slipping velocity v_S at any instant. Such velocity may, of course, change direction during collision, at any instant at which the slipping velocity given by Eq. (3.5) becomes zero.

4. MOTION AFTER INITIAL SLIPPING

In the event that slipping in one direction terminates, it can continue in the opposite direction. If slipping ends at a given instant, one has the conditions $v_{O1} = v_{O2} = 0$, so that the colliding bodies are either in a pure deformation or restitution phase. If the normal velocity v_{O3} has not reached the values given by Eq. (5.3) or (5.4), it will continue to increase its absolute value until such values are attained, and the acceleration of point O will be in the direction of $O\hat{x}_3$, so that from Eq. (2.4) it follows that

$$\partial \Phi / \partial F_1 = \partial \Phi / \partial F_2 = 0 \tag{4.1}$$

and the components F_1 and F_2 may be found as a function of F_3. Considering the known formula for Φ as given by Eq. (2.5), it is found that

$$F_1 = a_{13} F_3 , \; F_2 = a_{23} F_3 \tag{4.2}$$

where

$$\begin{aligned} a_{13} &= \frac{(1+S_{22})S_{13} - S_{12}S_{23}}{S_{12}^2 - (1+S_{11})(1+S_{22})} \\ a_{13} &= \frac{(1+S_{11})S_{23} - S_{12}S_{13}}{S_{12}^2 - (1+S_{11})(1+S_{22})} \end{aligned} \tag{4.3}$$

giving the value of the tangential reactions through the end of collision. Condition for not slipping, according to Coulomb's Law is $F_1^2 + F_2^2 < \mu^2 F_3^2$, so that one has to verify the condition

$$a_{13}^2 + a_{23}^2 < \mu^2 \tag{4.4}$$

which depends only on the position of the center of mass of any of the two colliding bodies, their inertia matrix and friction coefficient μ. If condition expressed by Eq. (4.4) is not satisfied, slipping will not terminate but will change direction. At this moment, the velocity of point O will be in the direction of the $O\hat{x}_3$ axis, since the tangential components are zero and the following slipping velocity $\vec{v}_S$ will be in the opposite direction of the friction force $\vec{F}_S = \vec{F} - F_3 \hat{x}_3$.

5. END OF COLLISION

Considering collision between perfect elastic bodies (restitution coefficient equal to unit), according to Newton's Law collision will terminate by the inversion of the normal velocity, that is when the accumulated elastic internal energy is returned to the bodies. If the collision is totally inelastic (restitution coefficient equal to zero), also according to Newton's Law, collision will terminates when that velocity reaches the same value for both bodies, zero in case of collision of a body against a fixed obstacle. In a particular case this is visualized in Fig. 1 below. In case of a perfect elastic collision the total work done by the normal force during collision is zero, that is the

energy input during deformation is equal to the energy output during restitution, this work being performed solely by the normal component F_3 of the contact reaction at point ^{o}
This is expressed by condition

$$\int_{t_0}^{t_1} F_3 \, v_{O3} \, dt = 0 \tag{5.1}$$

On the other end, Eqs. (2.3) and (2.5) give

$$\vec{F} = (I - S)^{-1} \dot{\vec{v}}_O = Y \, \dot{\vec{v}}_O \tag{5.2}$$

the inverse matrix existing in all cases except when all eigenvalues of S are equal to unit, a singular case to be excluded. From Eq. (5.2) it follows that $F_3 = \sum_k Y_{3\,k} \dot{v}_{O\,k}$ so that Eq. (5.1) becomes

$$\int_{t_0}^{t_1} \sum_k (Y_{3\,k} dv_{O\,k}) v_{O3} = 0 \tag{5.3}$$

In a head-on collision, Y_{31} and Y_{32} are zero, resulting the well known condition of total reversal of the normal velocity of the contact point O, i.e. $v_{O3}(t_1) = - v_{O3}(t_0)$.

6. EXAMPLES

6.1. Disk hitting a step

A simple application of the theory described in this paper is the case of a rough disk rolling, initially without slipping, over a horizontal table, and hitting a vertical step at his corner. It is supposed that the collision be perfectly elastic, the mass being unitary, radius R, central moment of inertia J and initial angular velocity ω_0. With notations already introduced, one has

$$\dot{\vec{v}}_O = (I - S)\vec{F} \tag{6.1.1}$$

where the problem is essentially two-dimensional, so that

$$v_{O1}(0) = -R\omega_0 \ sin\vartheta$$
$$v_{02}(0) = R\omega_0 \cos\vartheta$$
$$F_1 = \text{normal component of collision force}$$
$$F_2 = \text{tangential component of collision force}$$
$$I = \text{identity matrix } 2\times 2$$
$$S_{11} = 1, \ \ S_{22} = -R^2 / J$$

Vector $\dot{\vec{v}}_0$ is parallel to vector $(I - S)\vec{F}$ so that, under slipping conditions, one has the relation

$$\frac{\dot{v}_{O2}}{\dot{v}_{O1}} = \frac{(1 - S_{22})F_2}{F_1} = (1 - S_{22}) \ \mu \tag{6.1.2}$$

It follows that, since the ratio is constant, the finite increments in speed during slipping at collision are related by

$$\frac{\Delta v_{O2}}{\Delta v_{O1}} = (1 - S_{22})\ \mu \tag{6.1.3}$$

from where one finds at any given time, during collision and slipping, the velocity of the impacting point O as given by

$$\begin{aligned} v_{O1} &= v_{O1}(0) + \Delta v_{O1} \\ v_{O2} &= v_{O2}(0) + \Delta v_{O2} = v_{O2}(0) + (1 - S_{22})\ \mu\ \Delta v_{O1} \end{aligned} \tag{6.1.4}$$

If slipping stops, the tangential velocity of the contact point is zero, and in this case

$$v_{O1} = v_{O1}(0) + \Delta v_{O1} = v_{O1}(0) - \frac{v_{O2}(0)}{(1 - S_{22})\ \mu} \tag{6.1.5}$$

which is the velocity of point O at the end of slipping. If the velocity v_{O1} reaches value zero before the end of slipping, then $\Delta v_{O1} = -v_{O1}(0)$ and $v_{O2} = v_{O2}(0) - (1 - S_{22})\ \mu\ \ v_{O1}(0)$. Several other combinations are possible, by considering all possible cases v_{O1} = or < or > 0. In case of an elastic collision, the velocity with which point O emerges from collision along the normal direction, in this case line OG, is equal to a factor e (restitution coefficient) times the initial velocity, that is

$$v_{O1}(final) = -e v_{O1}(0) = e\omega_0\ R\ sin\vartheta \tag{6.1.6}$$

Under this assumption, the foregoing relations allow the determination of the final velocity along the tangential direction, indicating whether slipping ends before the end of collision or not.

6.2. Two rough equal spheres rolling over a horizontal plane

Two spheres of equal radius R and equal unitary masses, with angular velocities ω_1 and ω_2, impinge one upon the other in a head-on collision. Let the line joining G_1 and G_2 be taken as the x-axis, with origin at O_1, the point of sphere (1) colliding with point O_2 of sphere (2). Let the y-axis be directed in the upward direction. The initial velocities of O_1 and O_2 are given by

$$\begin{aligned} \vec{v}_{O_1}(0) &= (v_{G_1 1}(0),\ -R\omega_1) \\ \vec{v}_{O_2}(0) &= (v_{G_2 1}(0),\ -R\omega_2) \end{aligned} \tag{6.2.1}$$

In this case one has to consider the relative velocity of the points of contact, say of point O_1 relative to point O_2, that is,

$$\vec{v}_O = \vec{v}_{O_1} - \vec{v}_{O_2} = (v_{G_1 1} - v_{G_2 1}\ ,\ -R\omega_1 + R\omega_2) \tag{6.2.2}$$

The dynamical equations of the problem are

$$\dot{\vec{v}}_O = (2F_{11},\, 2\frac{R^2}{J}F_{12}) \qquad (6.2.3)$$

where we have used the law of action and reaction at the point of contact. From this point on the reasoning is similar to that used in the previous example.

6.3. Beghin's example (Chapter XV, n. 287, p. 481)

With obvious changes in notation, the problem of a falling rotating disk that hit a rough spike at point O, at distance a from G, may be represented by the following equations (we have considered M = 1):

$$\begin{aligned} &\vec{v}_O(0) = (-a\omega_0, -v_{vertical}) \\ &\dot{\vec{v}}_O = (\dot{v}_{O2}, \dot{v}_{O3}) = [(1-S_{22})F_2, (1-S_{33})F_3] \end{aligned} \qquad (6.3.1)$$

where $S_{22} = -2(a/R)^2, S_{33} = -4(a/R)^2$. Since slipping will certainly occur at the beginning of contact, unless an infinite friction is present, the evolution of the velocity of contact point O is given by the straight line

$$\frac{dv_{O2}}{dv_{O3}} = \frac{\Delta v_{O2}}{\Delta v_{O3}} = \mu\frac{1-S_{22}}{1-S_{33}} \qquad (6.3.2)$$

so that, at any given time during collision under slipping conditions, one has

$$v_{O2} = v_{O2}(0) + \mu\frac{1-S_{22}}{1-S_{33}}\Delta v_{O3} \qquad (6.3.3)$$

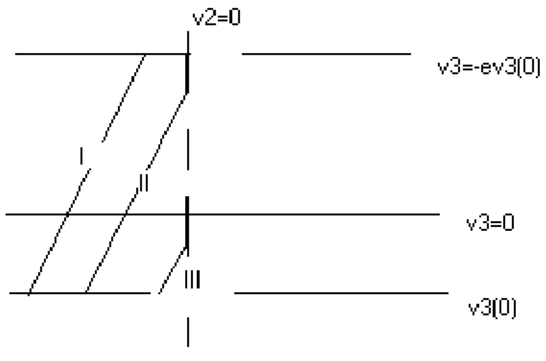

Figure 1
Three possible cases
of slipping

At the end of slipping, this velocity is zero, and the evolution depends on whether this happens before or after the normal velocity reaches the value dictated by Newton's law, that is

$v_{O3} = -ev_{O3}(0)$. Three possible situations are actually described geometrically by curves I, II, III drawn at page 481 of Beghin's work. Curves II and III (Figure 1) represent the condition $v_{O2} = 0$ before the end of collision, that is, the normal velocity at the end of slipping is given by

$$v_{O3} = v_{O3}(0) - \frac{1-S_{33}}{1-S_{22}} \frac{1}{\mu} \tag{6.3.4}$$

while curve I corresponds to slipping all through the duration of collision, with final normal velocity of point O satisfying Newton's Law.

7. CONCLUSION

A new mathematical formulation of the problem of rough solid bodies colliding with negligible deformations has been presented. A more comprehensive work has still to be developed, where the evolution of the rotational velocities should be determined. Another point to be explored should be the presence of pivoting friction, an important influence on these velocities. It is our view that a theory should be developed combining Routh's and Beghin's approaches.

REFERENCES

Beghin, H. 1952, "Cours de Mécanique Theorique et Appliqué", Tome I, Gauthierr-Villars, Paris

Darboux, M. 1880, "Études géométriques sur les percussions et les choc des corps", Bull. Sci. Math., 2nd Section, Vol. IV, 126-160

Derby, N. and Fuller, R. 1999, "Reality and Theory in a Collision", The Physics Teacher, 37, n.1, 24-27.

Giacaglia, G. E. O. 1967, "Problemas de Mecânica Geral", Vol. VI, Nobel, São Paulo

Levi-Civita, T. and Amaldi, U. 1937, "Lezioni di Meccanica Razionale", Vol. II, Part II, Nicola Zanichelli, Bologna (contém inúmeras referências sobre o assunto).

Mayer, A. 1902, "Über den Zusammstoff", Leipzig Berichte, 201-243

Pérès, J. 1926, "Choc en tenant compte du frottement", Nouv. Ann. Math., 5th Section, Vol. II, 98-108; 216-231

Routh, J. E. 1897, "The Elementary Part of a Treatise on the Dynamics of System of Rigid Bodies", Vol. 1, McMillan, London