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HEAT TRANSFER OF SIMPLIFIED PHAN-THIEN—TANNER FLUIDS IN PIPES AND CHANNELS

Paulo J. Oliveira
Departamento de Engenharia Electromecânica, Universidade da Beira Interior
Rua Marquês D'Ávila e Bolama, 6200 Covilhã, Portugal, pjpo@ubi.pt
Fernando T. Pinho
Centro de Estudos de Fenómenos de Transporte, DEMEGI, Faculdade de Engenharia, Universidade do Porto, Rua dos Bragas, 4099 Porto Codex, Portugal, fpinho@fe.up.pt

Abstract. *The rheology of some concentrated solutions of polymers and polymer melts is predicted adequately by the Phan-Thien—Tanner (PTT) constitutive equation (Larson, 1988, Quinzani et al, 1995). Such model fluids are frequently used to simulate real fluids encountered in industry, in processes involving high temperatures and heat transfer operations, and are also useful to assess the performance of numerical codes.*

For the simplified version of the PTT fluid model, an exact solution is derived for thermal and hydrodynamic fully-developed pipe and channel flows. The analysis considers a constant wall heat flux boundary condition and shows that fluid elasticity is responsible for an enhancement in heat transfer of at most 15.8% for the pipe flow and of 11.1% for the channel flow.

Keywords: *Viscoelastic, Phan-Thien—Tanner model, Heat transfer enhancement, Pipe flow*

1. INTRODUCTION

The Phan-Thien—Tanner (PTT) constitutive equation was derived from considerations of network theory by Phan-Thien and Tanner (1977) and is a simple model often used to simulate the rheological behaviour of polymer melts and concentrated solutions as in Quinzani et al (1995).

The simplified version of the PTT constitutive equation (SPTT) becomes

$$Y(\mathrm{tr}\tau, T)\tau + \lambda \overset{\nabla}{\tau} = 2\eta \mathbf{D} \tag{1}$$

where $\overset{\nabla}{\tau}$ stands for Oldroyd's upper convected derivative of the stress tensor τ

$$\overset{\nabla}{\tau} = \frac{D\mathbf{u}}{Dt} - \tau.\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}}.\tau \tag{2}$$

In order to simplify the analytical derivation we use th linearised stress coefficient without temperature dependence:

$$Y(tr\tau) = 1 + \frac{\varepsilon\lambda}{\eta}\mathrm{tr}\tau \tag{3}$$

In the above equations ε is a free parameter related to the extensional properties of te fluid; it imposes an upper limit to the elongational viscosity which is proportional to its inverse. When $\varepsilon = 0$, the upper-convected Maxwell model is recovered, which has an unbounded elongational viscosity in simple extensional flow. The parameter ε may have an influence on the shear properties as well, imparting shear-thinning to the fluid provided its value is not too small (Phan-Thien (1978) has shown no effect of ε when it is of the order of 10^{-2}). In the equations λ is a relaxation time, η is a viscosity coefficient equal to the product of the relaxation time by the relaxation modulus λG, $\mathbf{D}$ is the rate-of-strain tensor and $\mathrm{tr}\,\boldsymbol{\tau}$ is the trace of the stress tensor $\boldsymbol{\tau}$.

Viscoelastic fluid flow through pipes and channels is relevant in many industrial processing applications. In polymer processing, for example, melts flow in pipes at high temperatures before extruded and therefore knowledge of the temperature distribution and heat transfer coefficients is important for good design of polymer processing equipment. The current work contributes to a better understanding of the heat transfer phenomena in elastic liquids with view to improved engineering design.

The analytical hydrodynamic solution of the simplified PTT fluid flowing in a pipe and a channel has been obtained recently by Oliveira and Pinho (1999). Based on that solution we analyse now the corresponding heat transfer problem and derive the temperature distribution and heat transfer coefficient. It is remarked that for similar problems involving inelastic non-Newtonian fluids there is already a wealth of knowledge, in particular for fluids obeying the power law model (see, for instance, Irvine and Karni, 1987).

In the next section the problem is formulated and the solution of the corresponding hydrodynamic problem will be presented. Then, the analytical solution will be derived in detail for the heat transfer problem and the effect of the rheological parameters on the relevant heat transfer quantities will be discussed. The derivation and discussion of results will be carried out in detail for the pipe flow geometry only, whereas for the equivalent planar geometry the final results will be presented without further comment.

2. FORMULATION OF THE PROBLEM

The flow is considered to be fully-developed both thermally and hydrodynamically. It is also assumed that the flow is steady, laminar and has constant properties, i.e., no dependence of the fluid properties and model parameters on temperature will be considered. The boundary condition is that of an imposed heat flux at the pipe wall.

The energy equation to be solved in the axisymmetric case is

$$k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\rho c_p u\frac{\partial T}{\partial x} \tag{4}$$

where k, ρ and c_p stand for the thermal conductivity, density and specific heat, respectively. The temperature T varies radially (r) and axially (x) and u stands for the longitudinal velocity component. Effects of viscous dissipation are here neglected and will be investigated in future work.

The thermal boundary conditions are then

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0 \tag{5}$$

expressing axisymmetry and a constant heat flux at the wall

$$\left.-k\frac{\partial T}{\partial r}\right|_{r=R}=\dot{q}_w \tag{6}$$

The velocity profile required in Eq. (4) takes the form derived by Oliveira and Pinho (1999) and given in Eq. (7)

$$\frac{u}{\bar{u}}=2\frac{\overline{u_N}}{\bar{u}}\left[1-\left(\frac{r}{R}\right)^2\right]\left\{1+16\varepsilon De^2\left(\frac{\overline{u_N}}{\bar{u}}\right)^2\left[1+\left(\frac{r}{R}\right)^2\right]\right\} \tag{7}$$

Here the non-dimensional group $De \equiv \lambda \bar{u}/R$ is the Deborah number, a measure of the level of elasticity in the fluid. It is based on the cross-sectional average velocity $\bar{u}$ for the PTT fluid. $\overline{u_N}$ is the average velocity for a Newtonian fluid flowing under the same pressure gradient dp/dx

$$\overline{u_N} \equiv \frac{-(dp/dx)R^2}{8\eta} \tag{8}$$

and was shown to be given by:

$$\frac{\overline{u_N}}{\bar{u}} = \frac{432^{1/6}\left(\delta^{2/3} - 2^{2/3}\right)}{6b^{1/2}\delta^{1/3}} \text{ with } \delta = \left(3^3 b + 4\right)^{1/2} + 3^{3/2} b^{1/2} \text{ and } b = \frac{64}{3}\varepsilon De^2 \tag{9}$$

For the following derivations it will be advantageous to use a modification of b, which we will designate as a, which is defined below.

$$a \equiv 16\varepsilon De^2 \left(\frac{\overline{u_N}}{\bar{u}}\right)^2 \tag{10}$$

This non-dimensional parameter gives a measure of both the extensional (measured by ε) and the elastic (measured by De) properties of the fluid.

3. ANALYTICAL SOLUTION

Since the analytical solution for the velocity profile is known, the derivation of the heat transfer quantities follow the same steps as in the corresponding classical Newtonian case (see for instance Holman, 1981).

3.1 Pipe flow

The constant wall flux boundary condition (Eq. 6) implies that the cross-section average temperature ($\overline{T}$) must increase longitudinally at a constant rate. This, together with the condition of thermal fully-developed flow, implies a constant longitudinal gradient of temperature $\partial T/\partial x$.

The velocity from Eq. (7) is replaced in the energy Eq. (4) which can then be integrated a first time. The axisymmetry boundary condition (Eq. 5) is applied next to solve for the first constant of integration leading to the radial distribution of the gradient of temperature

$$\frac{\partial T}{\partial r} = \frac{2R\overline{u_N}}{\alpha}\frac{dT}{dx}\left[\frac{(1+a)}{2}\frac{r}{R} - \frac{1}{4}\left(\frac{r}{R}\right)^3 - \frac{a}{6}\left(\frac{r}{R}\right)^5\right] \tag{11}$$

where the thermal properties have been compacted into the definition of the thermal diffusivity

$$\alpha \equiv \frac{k}{\rho c_p}$$

This equation is now integrated a second time and the second boundary condition is applied indirectly. Instead of using immediately Eq. (6), it is more convenient to introduce now the centreline temperature T_c and to relate it at a later stage with the wall heat flux, by making use of the definitions of $\dot{q}_w$ and of the bulk temperature $\overline{T}$. Thus, the temperature distribution becomes

$$T - T_c = \frac{2\overline{u_N}R^2}{\alpha}\frac{dT}{dx}\left[\frac{1+a}{4}\left(\frac{r}{R}\right)^2 - \frac{1}{16}\left(\frac{r}{R}\right)^4 - \frac{a}{36}\left(\frac{r}{R}\right)^6\right] \tag{12}$$

The wall temperature is easily obtained from Eq. (12) by setting $T = T_w$ at $r/R = 1$ and is given by

$$T_w - T_c = \frac{\overline{u_N} R^2}{2\alpha} \frac{dT}{dx} \left[\frac{3}{4} + \frac{8}{9} a \right] \tag{13}$$

The heat transfer coefficient (h) for this forced convection flow is defined from the wall heat flux $\dot{q}_w$ as

$$\dot{q}_w \equiv h \left(\overline{T} - T_w \right) \tag{14}$$

where the cross-section average temperature is given by

$$\overline{T} \equiv \frac{\int_0^R 2\pi u T r dr}{\int_0^R 2\pi u r dr} \tag{15}$$

The denominator of Eq. (15) represents the volumetric flow rate $\pi R^2 \bar{u}$. Oliveira and Pinho (1999) give the following expression for the bulk velocity

$$\bar{u} = \overline{u_N} \left(1 + \frac{4}{3} a \right) \tag{16}$$

to be used in the ensueing analysis. Note, however, that a still depends on $\bar{u}$. Upon substitution, integration of Eq. (15) yields the bulk temperature

$$\overline{T} - T_c = \frac{\frac{\overline{u_N} R^2}{\alpha} \frac{dT}{dx} \left[\frac{13}{54} a^2 + \frac{17}{45} a + \frac{7}{48} \right]}{1 + \frac{4}{3} a} \tag{17}$$

The boundary condition (Eq. 6) is now introduced through the heat transfer coefficient

$$h = \frac{-k \left(\frac{\partial T}{\partial r} \right)_{r=R}}{\overline{T} - T_w} \tag{18-a}$$

which is calculated next and presented in non-dimensional form as a Nusselt number

$$Nu \equiv \frac{D_H h}{k} = \frac{2Rh}{k} = \frac{2R\dot{q}_w}{k \left(\overline{T} - T_w \right)} \tag{18-b}$$

After performing the necessary substitutions in Eq. (18-b) we obtain:

$$Nu = \frac{\left[1 + \frac{4}{3} a \right]^2}{\left[\frac{19}{54} a^2 + \frac{17}{30} a + \frac{11}{48} \right]} \tag{19}$$

Equation (19) reduces to the well-known Newtonian solution of Nu=4.364 (Holman, 1981)in the absence of either elasticity or extensional effects (both leading to $a = 0$). For $a \to 0$, it gives $Nu = 5.053$.

The above equation for the temperature profile *T(r)* can also be casted into a non-dimensional form using the usual definition

$$\theta(r) \equiv \frac{T(r) - T_w}{\overline{T} - T_w} \tag{20}$$

and the resulting profile is:

$$\theta(r) = \frac{-2\left[1+\frac{4}{3}a\right]\left\{\frac{1+a}{4}\left(\frac{r}{R}\right)^2 - \frac{1}{16}\left(\frac{r}{R}\right)^4 - \frac{a}{36}\left(\frac{r}{R}\right)^6 - \frac{3}{16} - \frac{8}{36}a\right\}}{\left[\frac{19}{54}a^2 + \frac{17}{30}a + \frac{11}{48}\right]} \tag{21}$$

However, as will be shown in the discussion section, the dimensionless temperature defined by Eq. (21) is not so illustrative and improved understanding of the phenomena involved will require a different scaling and the following expressions for the various relevant temperature differences:

$$\frac{\left(T(r)-T_w\right)k}{\dot{q}_w R} = \frac{-1}{1+\frac{4}{3}a}\left\{(1+a)\left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4 - \frac{a}{9}\left(\frac{r}{R}\right)^6\right\} \tag{22}$$

$$\frac{\left(T_w - T_c\right)k}{\dot{q}_w R} = -\frac{\frac{3}{4}+\frac{8}{9}a}{1+\frac{4}{3}a} \tag{23}$$

$$\frac{\left(\overline{T} - T_c\right)k}{\dot{q}_w R} = -\frac{2\left[\frac{13}{54}a^2 + \frac{17}{45}a + \frac{7}{48}\right]}{\left[1+\frac{4}{3}a\right]^2} \tag{24}$$

3.2 Channel flow

For the channel flow the analytical derivation is similar and is based on the hydrodynamic solution of Oliveira and Pinho (1999). The analysis starts with the set of equations corresponding to Eqs. (4) to (10) for the plane channel and the results are presented below without any other details. The tranverse coordinate is *y* and the channel half-width is equal to *H*.

For this flow geometry we use

$$a \equiv 9\varepsilon De^2\left(\frac{\overline{u_N}}{\overline{u}}\right)^2 \tag{25}$$

and we obtain the following results. The transverse distribution of temperature is

$$T - T_c = \frac{3\overline{u_N}H^2}{2\alpha}\frac{dT}{dx}\left[\frac{1+a}{2}\left(\frac{y}{H}\right)^2 - \frac{1}{12}\left(\frac{y}{H}\right)^4 - \frac{a}{30}\left(\frac{y}{H}\right)^6\right] \tag{26-a}$$

and in non-dimensional form

$$\theta(y) = -\frac{\left[1+\frac{6}{5}a\right]\left\{\frac{1+a}{2}\left(\frac{y}{H}\right)^2 - \frac{1}{12}\left(\frac{y}{H}\right)^4 - \frac{a}{30}\left(\frac{y}{H}\right)^6 - \frac{5}{12} - \frac{7}{15}a\right\}}{\left[\frac{808}{1925}a^2 + \frac{232}{315}a + \frac{102}{315}\right]} \tag{26-b}$$

From Eq. (26-a) we get the wall temperature by setting $y/H=1$

$$T_w - T_c = \frac{3\overline{u_N}H^2}{4\alpha}\frac{dT}{dx}\left[\frac{5}{6}+\frac{14}{15}a\right] \tag{27}$$

The cross-sectional average temperature is given by

$$\overline{T} - T_c = \frac{\dfrac{9\overline{u_N}H^2}{20\alpha}\dfrac{dT}{dx}\left[\dfrac{108}{231}a^2 + \dfrac{145}{189}a + \dfrac{13}{42}\right]}{1 + \dfrac{6}{5}a} \tag{28}$$

The final expression for the Nusselt number $\left(Nu \equiv \dfrac{4Hh}{k}\right)$ becomes

$$Nu = \frac{4\left[1 + \dfrac{6}{5}a\right]^2}{\left[\dfrac{1212}{1925}a^2 + \dfrac{116}{105}a + \dfrac{17}{35}\right]} \tag{29}$$

Similarly to the pipe flow, Eq. (29) reduces to the well known Newtonian value ($a = 0$) of $Nu = 8.235$ and tends to 9.149 when $a \to \infty$.

4. DISCUSSION OF RESULTS

In Fig. 1 the Nusselt number of Eq. (22) is plotted as a function of the Deborah number and parameter ε. The Nusselt number varies between two asymptotic values: the Newtonian value of 4.364 in the limit of low elasticity and ε, and 5.053 which is the limit of Eq. (19) as $a \to \infty$. The variation corresponds to an increase of at most 15.8% relative to the Newtonian value.

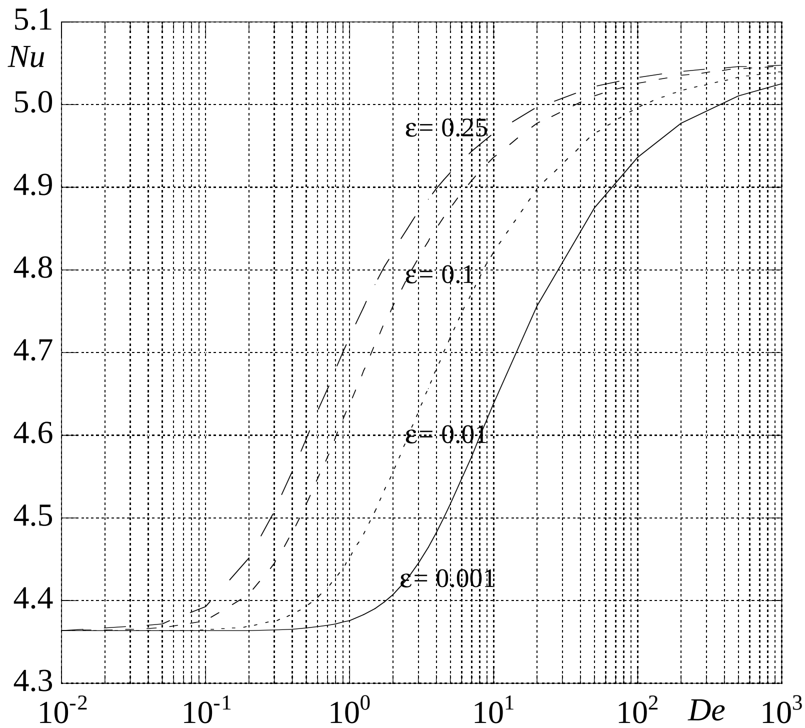

Figure 1) Variation of the Nusselt number as a function of the Deborah number and ε.

The effect of the elongation-related parameter ε is to anticipate the transition to lower values of the Deborah number. It is important to note however, that as $\varepsilon \to 0$ the Nusselt number reamins constant at the Newtonian value of 4.364 for all De. The heat transfer enhancement by viscoelasticity observed in Fig. 1 is related to the way the velocity profile is modified when ε and De are increased. Oliveira and Pinho (1999) have shown that increasing values of $\sqrt{\varepsilon}De$ impart shear-thinning behaviour to the fluid with flatter velocity profiles in the core and higher wall shear rates, as can be assessed in the velocity plot of Fig. 2.

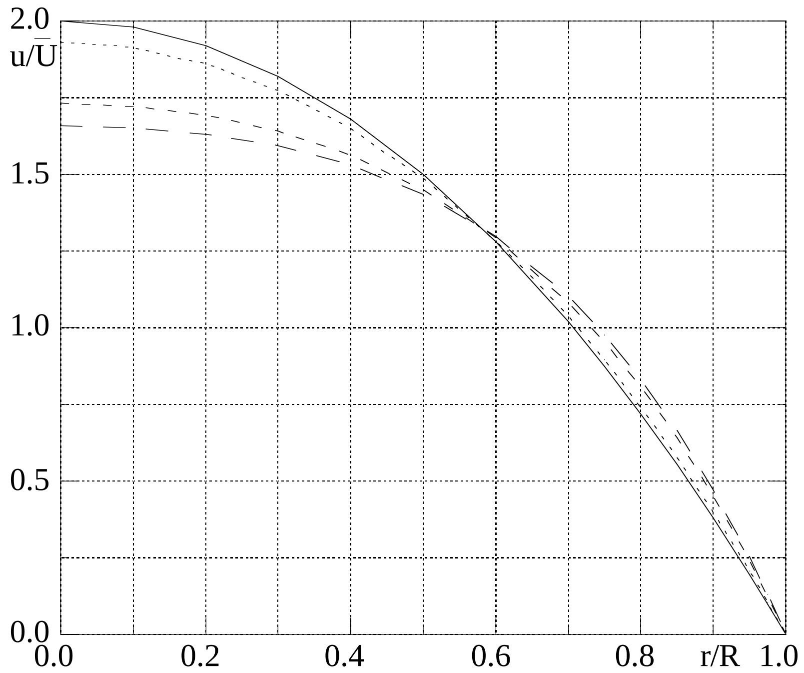

Figure 2- Velocity plot of the linear PTT fluid in pipe flow as a function of the dimensionless group $\sqrt{\varepsilon}De$. (solid line: parabolic profile; increased dashed lines: $\sqrt{\varepsilon}De$= 0.1, 0.5 and 1.0).

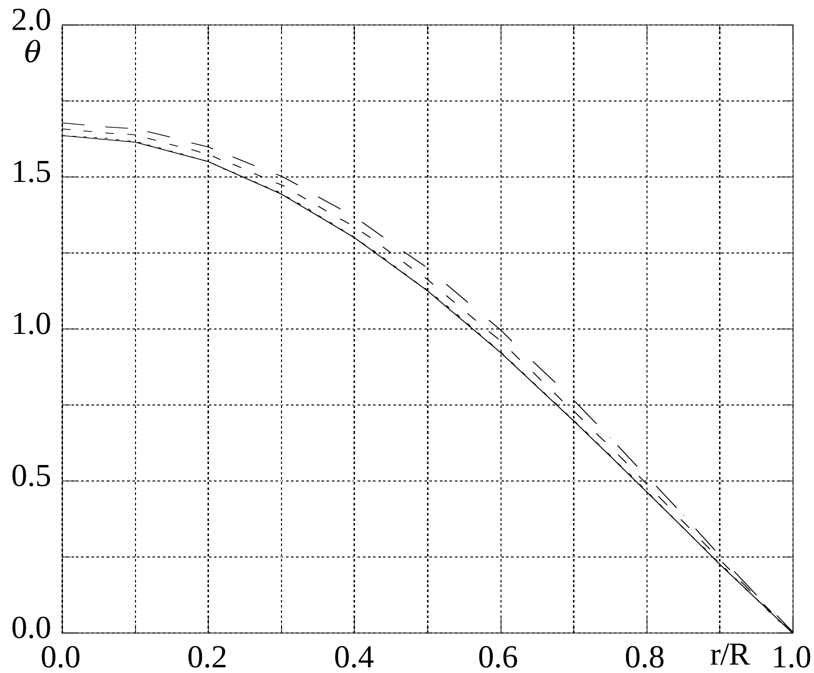

Figure 3) Effect of *De* on the radial variation of the standard normalised temperature θ for $\varepsilon = 0.1$. (solid line: *De*= 0; increased dashed lines: *De*= 0.1, 1, 10)

Plots of the dimensionless temperature θ are presented in Fig. 3 and 4 to assess the effects of De and ε. The standard way of making temperature non-dimensional, based on Eq.

(20) and shown in Figs. (3) and (4), is not appropriate for the situation of imposed heat-flux because the temperature scale in denominator ($\Delta T = \overline{T} - T_w$) also varies with the relevant parameters and may lead to misinterpretation of the corresponding variation of T. For example, in both figures the range of non-dimensional temperatures θ within the pipe increases with ε and *De* when in fact the range of dimensional temperatures T is reduced due to an increased heat transfer coefficient, for an imposed constant heat flux.

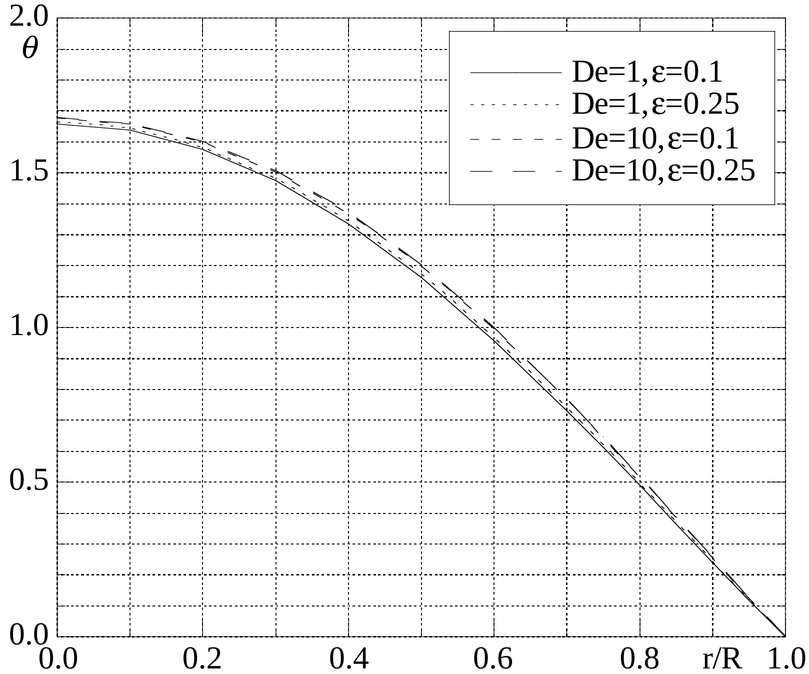

Figure 4) Effect of ε and *De* on the radial variation of the standard normalised temperature θ.

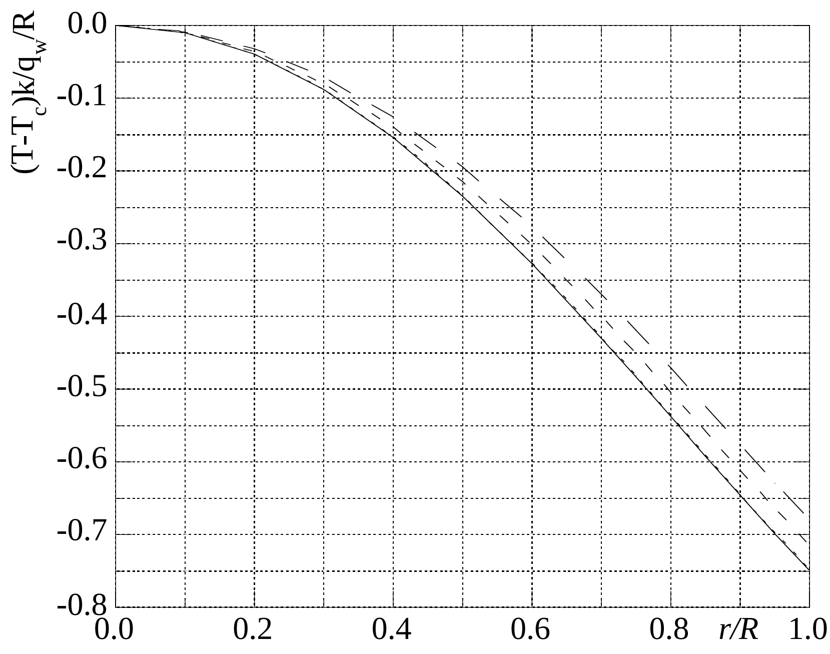

Figure 5) Radial profiles of the normalised temperature as a function of Deborah for ε= 0.1. (solid line: *De*= 0; increased dashed lines: *De*= 0.1, 1, 10)

Another apparent contradiction concerns the temperature gradient at the wall. Whereas the gradient of the non-dimensional temperature q is seen to increase with fluid elasticity, the gradient of the temperature T must remain constant for a given $\dot{q}_w$. In fact, for a given $\dot{q}_w$, the unknown of the problem is ΔT and it is thus more convenient to define a fixed temperature scale for normalisation that we take as $\dot{q}_w R/k$. Now, with this normalisation of the temperature the slope of the curves near the wall must be equal to -1 for all cases, as is apparent in Fig. 5.

Fig. 5 shows one set of temperature profiles made non-dimensional with this fixed temperature scale for various Deborah numbers, at ε=0.1. This plot makes clear the forementioned effect of increased elasticity in reducing the range of temperature variation accross the pipe section, thus leading to somewhat higher Nusselt numbers (cf. Eq. 18-b).

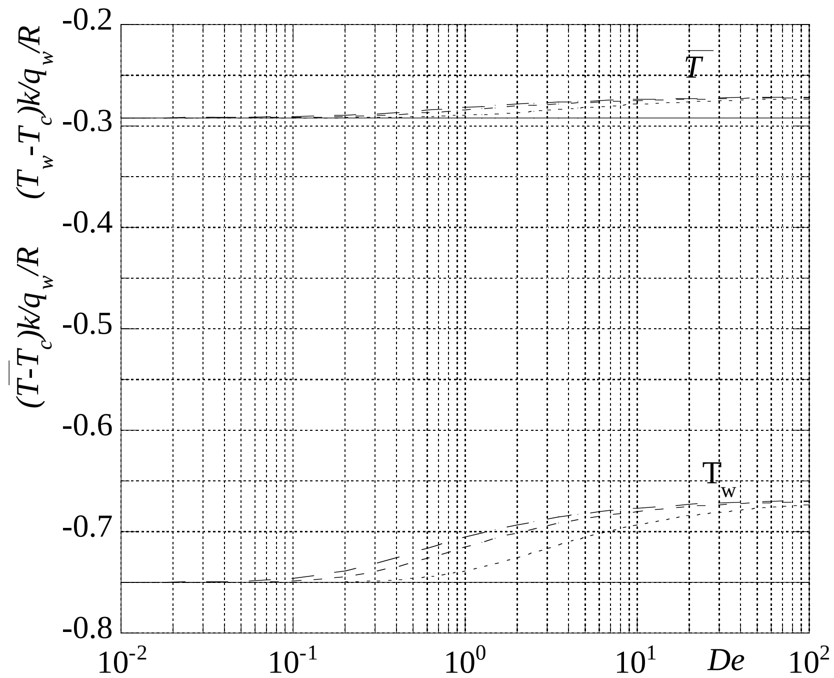

Figure 6) Variation of the normalised bulk and wall temperatures with the Deborah number for the cases without (*De*= 0, solid) and with (*De*= 5, dashed) elasticity. Increased dash lines: ε = 0.01, 0.1 and 0.25.

Finally, in Fig. 6 are plotted curves of $\frac{(T_w - T_c)k}{\dot{q}_w R}$ from equation (23) and $\frac{(\bar{T} - T_c)k}{\dot{q}_w R}$ from Eq. (24) as a function of *De*, for various values of ε. Both quantities show the same asymptotic behaviour of the Nusselt number plots. The effect of elasticity is felt more intensely by the wall temperature than on the bulk temperature since the thermal resistance is lower for the fluid layer closer to the wall and also because elasticity increases the proportion of the total flow rate that flows in the vicinity of the wall.

Note that the combined effects of ε and *De* are felt via a and they actually act as a single non-dimensional number $\sqrt{\varepsilon} De$, which implies that variations of *De* have a stronger impact than similar variations of ε.

For the channel flow similar conclusions could be drawn from a similar study.

5. CONCLUSIONS

Temperature distributions and heat transfer coefficients were obtained analytically for fully-developed pipe and channel flows of a simplified Phan-Thien—Tanner fluid when the stress coefficient assumed a linear form and the effect of temperature variations on the material parameters was neglected. Viscous dissipation was not accounted for.

An increase of fluid elasticity (*De*) and/or an increase of ε resulted in enhanced heat transfer coefficients with the Nusselt number limited by two asymptotic values: for low values of $\sqrt{\varepsilon}De$ the Nusselt number takes the Newtonian value of 4.364 and at high elasticity it tends to 5.053, a maximum increase of 15.8%. The solution shows that the enhanced heat transfer coefficient results from a reduction in the range of temperatures within the pipe, with a stronger impact felt in the wall region. Thus, in order to transfer a certain amout of heat the viscoelastic fluid will require a smaller difference between the wall and the bulk temperatures than the Newtonian fluid. These effects are the consequence of $\sqrt{\varepsilon}De$ in imparting a shear-thinning behaviour to the velocity profile.

Purely elastic fluids like the upper convected Maxwell fluid ($\varepsilon = 0$, $De \neq 0$) have heat transfer characteristics equal to those of Newtonian fluids.

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