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ANALYSIS OF STOCHASTIC STRUCTURES BY PERTURBATION METHOD

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Abstract. *In this paper we analyze the modal characteristics of structures including random parameters. We compute the mean value and the standard deviation of the eigenvalues and eigenvectors and also of the transfer functions. The stochastic finite element method base on Perturbation technique, will be used to do so. The stochastic finite element method attempt to combine the finite element analysis and the stochastic analysis. Limits of perturbation approach are studied in terms of sensitivity of the solution (modal parameters and transfer functions) versus the random parameters values. A Monte Carlo Simulation is used to validate our results. In order to reduce the number of degrees of freedom of the model and the time consuming, a modal synthesis approach extended to the case of structures defined by stochastic parameters is used. We will emphasize the efficiency of this procedure in the estimation of the mean value and the standard deviation of the modal parameters.*

Keywords: *Modal Synthesis ; Stochastic Parameters ; Perturbation*

1. INTRODUCTION

The extension of the finite element method to take in account the uncertainties in the geometry or material properties of a structure, as well as the applied loads, is spelled Stochastic Finite Element Method. This field has recently become an active area of research because of perception that in some structures the response is strongly sensitive to the small random variation in material properties or geometry of the structure. Eccentricity in cross-section, differences of mass density and/or Young's modulus are examples of randomness of structural engineering problems. Such uncertainties are usually spatially distributed over the region of the structure and should be modeled as random fields. Several methodologies can be adopted to evaluate structural response uncertainties. Early applications used the Monte Carlo simulation (Astill *et al.*, 1972), which computes the responses for a (large) set of random numbers representing the uncertainties. Such a method is time consuming and needs a lot of CPU. Later, the Taylor series expansions, sensitivity vectors methods and perturbation methods were used to compute the second-moment statistics of response quantities in structural applications. These methods are mathematically identical to the second order of perturbation method (Benaroya & Rehak, 1988). The basic idea of the second-moment analysis of stochastic systems by perturbation method, is to expand, via Taylor series, all the stochastic field variables about the mean values of random variables, to retain only up to second-order terms. The output expectations and cross-covariances are obtained from the inputs expectations and cross-covariances. This method is much faster than the Monte Carlo one, but it is limiting of the low values of dispersion (Kleiber & Hien, 1992). However, the development of Taylor of the random fields discretized by finite element makes increase the number of equations to solve. If we are interested in solving industrial problems; then this technique is too much time consuming. In order to reduce the number of degrees of freedom of the model and the time consuming, the Component Modal Synthesis methods are frequently employed in structural dynamics. These methods, presented in several reviews (Craig-Jr, 1987, by example) and numerous papers, apply the sub-structuring techniques to reduce the size of the problem. The Stochastic Component Modal Synthesis method, presented in this paper attempt to reduce the size of the problem using a modal synthesis approach extended to the case of structures defined by stochastic parameters.

2. PERTURBATION TECHNIQUE

We consider a conservative dynamic system discretized by a finite element mesh, with random design parameters. For example, we take a bar subjected to axial vibration divided into two parts with different random Young's moduli. For simplification, the Young's moduli are time and spatial invariant. It is also assumed that all the others parameters are deterministic. So, we have two statistically independent Gaussian random variables ($\widetilde{N} = 2$) "E_1" and "E_2".

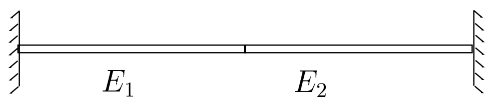


Figure 1- Clamped-clamped bar with two random Young's moduli statistically independent.

Since the mass matrix is independent of the random variables, the eigenproblem of the studied bar is given by: (with $N =$ number of DOF)

$$([K(E_1, E_2)] - \omega_j^2(E_1, E_2)[M])\{\phi_j(E_1, E_2)\} = \{0\} \qquad j = 1, 2, ...N \tag{1}$$

By employing the Taylor series, we expand the stiffness matrix and eigenmodes retaining up to the second-order terms: (with $\widetilde{N} =$ number of random variables)

$$\left[K(E)\right] = [K]^0 + \sum_{\rho=1}^{\widetilde{N}} [K]^I_{E_\rho} \Delta E_\rho + \frac{1}{2} \sum_{\rho,\sigma=1}^{\widetilde{N}} [K]^{II}_{E_\rho E_\sigma} \Delta E_\rho \Delta E_\sigma \tag{2}$$

$$\left[\omega^2(E)\right] = [\omega^2]^0 + \sum_{\rho=1}^{\widetilde{N}} [\omega^2]^I_{E_\rho} \Delta E_\rho + \frac{1}{2} \sum_{\rho,\sigma=1}^{\widetilde{N}} [\omega^2]^{II}_{E_\rho E_\sigma} \Delta E_\rho \Delta E_\sigma \tag{3}$$

$$\left[\Phi(E)\right] = [\Phi]^0 + \sum_{\rho=1}^{\widetilde{N}} [\Phi]^I_{E_\rho} \Delta E_\rho + \frac{1}{2} \sum_{\rho,\sigma=1}^{\widetilde{N}} [\Phi]^{II}_{E_\rho E_\sigma} \Delta E_\rho \Delta E_\sigma \tag{4}$$

By simplicity we use the following notation:

$$[\alpha]^{I}_{E_\rho} = \frac{\partial}{\partial E_\rho}[\alpha]\,; \qquad [\alpha]^{II}_{E_\rho E_\sigma} = \frac{\partial^2}{\partial E_\rho \partial E_\sigma}[\alpha]\,;$$

$[\alpha]^{0} = [\alpha]$ evaluated at mean value of the random variables.

To a bar, the stiffness matrix second derivative versus the Young's module is zero, then the second term in the Eq. (2) vanish.

By Substituting the expansions (2) to (4) into Eq. (1), multiplying the second-order terms by the probability density function of the random variables and integrating over the domain of these variables, we get the equations for the stochastic eigenproblem. By collecting terms of equal orders we have:

Zeroth-order: One system of "N" equations

$$\left([K]^{0} - (\omega_j^2)^{0}[M]\right)\{\phi_j\}^{0} = \{0\} \qquad j = 1, 2, ...N \tag{5}$$

First-order : "$\widetilde{N}$" systems of "N" equations

$$\left([K]^{0} - (\omega_j^2)^{0}[M]\right)\{\phi_j\}^{I}_{E_\rho} = -\left([K]^{I}_{E_\rho} - (\omega_j^2)^{I}_{E_\rho}[M]\right)\{\phi_j\}^{0}$$
$$j = 1, 2, ...N \text{ et } \rho = 1, 2, ...\widetilde{N} \tag{6}$$

Second-order : One system of "N" equations

$$\left([K]^{0} - (\omega_j^2)^{0}[M]\right)\{\phi_j\}^{(2)} = -\sum_{\rho,\sigma=1}^{\widetilde{N}} \left(-(\omega_j^2)^{II}_{E_\rho E_\sigma}[M]\{\phi_j\}^{0}\right.$$
$$\left. + 2\left([K]^{I}_{E_\rho} - (\omega_j^2)^{I}_{E_\rho}[M]\right)\{\phi_j\}^{I}_{E_\sigma}\right)\mathbf{Cov}(E_\rho, E_\sigma) \qquad j = 1, 2, ...N \tag{7}$$

with:

$$\{\phi_j\}^{(2)} = \sum_{\rho,\sigma=1}^{\widetilde{N}} \{\phi_j\}^{II}_{E_\rho E_\sigma}\mathbf{Cov}(E_\rho, E_\sigma) \tag{8}$$

Where "$\mathbf{Cov}(E_\rho, E_\sigma)$" is the "$E_\rho$" and "$E_\sigma$" cross-covariance $(\rho, \sigma = 1, 2, \ldots \widetilde{N})$.

In the case of the variables are statistically independents and adopting the solution technique proposed by Kleiber & Hien (1992), we get these expressions for the mean and variance values:

$$\boldsymbol{\mu}(\omega_j^2) = (\omega_j^2)^{0} + \frac{1}{2}(\omega_j^2)^{(2)} \tag{9}$$

$$\boldsymbol{Var}(\omega_j^2) = \sum_{\rho=1}^{\widetilde{N}} \left((\omega_j^2)^{I}_{E_\rho}\right)^2 \cdot \boldsymbol{Var}(E_\rho) \tag{10}$$

$$\boldsymbol{\mu}\{\phi_j\} = \{\phi_j\}^{0} + \frac{1}{2}\{\phi_j\}^{(2)} \tag{11}$$

$$\boldsymbol{Var}\{\phi_j\} = \sum_{\rho=1}^{\widetilde{N}} \left(\{\phi_j\}^{I}_{E_\rho}\right)^2 \cdot \boldsymbol{Var}(E_\rho) \tag{12}$$

with the partial derivatives of the eigenvalues defined as:

$$(\omega_j^2)^I_{E_\rho} = \{\phi_j^T\}^0 [K]^I_{E_\rho} \{\phi_j\}^0 \tag{13}$$

$$(\omega_j^2)^{(2)} = 2\sum_{\rho=1}^{\widetilde{N}} \{\phi_j^T\}^0 \Big([K]^I_{E_\rho} - (\omega_j^2)^I_{E_\rho}[M]\Big)\{\phi_j\}^I_{E_\rho} \cdot \boldsymbol{Var}(E_\rho) \tag{14}$$

The eigenvectors derivatives are defined as a linear combination of the ”m” first zeroth-order eigenvectors (Fox & Kapoor, 1968):

$$\{\phi_j\}^I_{E_\rho} = \sum_{\substack{l=1 \\ l\neq j}}^{m} \left(\frac{\{\phi_l^T\}^0 \{F_j\}^I_{E_\rho}}{\omega_j^2 - \omega_l^2}\right) \{\phi_l\}^0 \qquad j = 1, 2, ...N \text{ and } \rho = 1, 2, \ldots \widetilde{N} \tag{15}$$

and:

$$\{\phi_j\}^{(2)} = \sum_{\substack{l=1 \\ l\neq j}}^{m} \left(\frac{\{\phi_l^T\}^0 \{F_i\}^{(2)}}{\omega_j^2 - \omega_l^2}\right) \{\phi_l\}^0 \qquad j = 1, 2, ...N \text{ and } \rho = 1, 2, \ldots \widetilde{N} \tag{16}$$

where:

$$\{F_j\}^I_{E_\rho} = -\Big([K]^I_{E_\rho} - (\omega_j^2)^I_{E_\rho}[M]\Big)\{\phi_j\}^0 \quad j = 1, 2, ...N \tag{17}$$

$$\{F_j\}^{(2)} = (\omega_j^2)^{(2)}[M]\{\phi_j\}^0 - \sum_{\rho=1}^{2} 2\left([K]^I_{E_\rho} - (\omega_j^2)^I_{E_\rho}[M]\right)\{\phi_j\}^I_{E_\rho} \cdot \boldsymbol{Var}(E_\rho) \qquad j = 1, 2, ...N \tag{18}$$

It should be emphasized that the occurence of the double sums in the Eq. (7) it makes with that the formulation described above requires only one second-order system to be solved instead of $\widetilde{N}(\widetilde{N}+1)/2$ systems apparently required if we look at the number of variables aleatoires and the symmetry of the problem.

3. THE FREQUENCY RESPONSE FUNCTION TAYLOR’S EXPANSION

The Frequency Response Function (FRF) is a usual representation of the dynamic behavior of systems, it is also a current form to identify them. Thus, in the study of the stochastic structures it is interessant to have the stochastic expression of the FRF.

If we consider hysteretic damping into the bar, we have the following expression for the FRF matrix components in modal expression:

$$H_{kj}(\omega) = \sum_{r=1}^{N} \frac{\phi_{kr}\phi_{rj}}{(1+i\eta)\omega_r^2 - \omega^2} \tag{19}$$

By way of simplification, we take the damping ”η” as being independent of the random variables. In this case, the Taylor series expansion terms of the FRF matrix components are:

$$(H_{kj}(\omega))^0 = \sum_{r=1}^{N} \frac{\phi_{kr}^0 \phi_{rj}^0}{(1+i\eta)(\omega^2)_r^0 - \omega^2} \tag{20}$$

$$
(H_{kj}(\omega))^{I}_{E_\rho} = \sum_{r=1}^{N} \left(\frac{(\phi_{kr})^{I}_{E_\rho} (\phi_{rj})^{0} + (\phi_{kr})^{0} (\phi_{rj})^{I}_{E_\rho}}{(1+i\eta)(\omega_r^2)^0 - \omega^2} + \right.
$$
$$
\left. -(1+i\eta) \frac{(\phi_{kr})^{0} (\phi_{rj})^{0} (\omega_r^2)^{I}_{E_\rho}}{((1+i\eta)(\omega_r^2)^0 - \omega^2)^2} \right) \quad (21)
$$

$$
(H_{kj}(\omega))^{II}_{E_\rho E_\sigma} = \sum_{r=1}^{N} \left(\frac{(\phi_{kr})^{II}_{E_\rho E_\sigma} (\phi_{rj})^{0} + 2(\phi_{kr})^{I}_{E_\rho} (\phi_{rj})^{I}_{E_\sigma} + (\phi_{kr})^{0} (\phi_{rj})^{II}_{E_\rho E_\sigma}}{(1+i\eta)(\omega_r^2)^0 - \omega^2} + \right.
$$
$$
-(1+i\eta) \frac{\left((\phi_{kr})^{I}_{E_\rho} (\phi_{rj})^{0} + (\phi_{kr})^{0} (\phi_{rj})^{I}_{E_\rho}\right) (\omega_r^2)^{I}_{E_\sigma}}{((1+i\eta)(\omega_r^2)^0 - \omega^2)^2} +
$$
$$
\left. -(1+i\eta) \frac{(\phi_{kr})^{0} (\phi_{rj})^{0} (\omega_r^2)^{II}_{E_\rho E_\sigma}}{((1+i\eta)(\omega_r^2)^0 - \omega^2)^2} + 2(1+i\eta)^2 \frac{(\phi_{kr})^{0} (\phi_{rj})^{0} (\omega_r^2)^{I}_{E_\rho} (\omega_r^2)^{I}_{E_\sigma}}{((1+i\eta)(\omega_r^2)^0 - \omega^2)^3} \right) \quad (22)
$$

The expressions for the mean and variance values to the FRF matrix component ”kj” are:

$$
\boldsymbol{\mu}(H_{kj}(\omega)) = (H_{kj}(\omega))^{0} + \frac{1}{2} \sum_{\rho,\sigma=1}^{\widetilde{N}} (H_{kj}(\omega))^{II}_{E_\rho E_\sigma} \mathbf{Cov}(E_\rho, E_\sigma) \quad (23)
$$

$$
\mathbf{Cov}(H_{kj}(\omega)) = \sum_{\rho,\sigma=1}^{\widetilde{N}} (H_{kj}(\omega))^{I}_{E_\rho} (H_{kj}(\omega))^{I}_{E_\sigma} \mathbf{Cov}(E_\rho, E_\sigma) \quad (24)
$$

In our case, the variables are statistically independents and the Eq. (23) and Eq. (24) are simplified.

4. STOCHASTIC FIXED INTERFACE COMPONENT SYNTHESIS METHOD

For large industrial problems, substructuration is an effective method for reducing the number of DOF, and then the size of problems and time computation. Moreover, when coupled with the perturbation method, substructuration enables the use of less random variables for each substructure. We use the Craig and Bampton method here, because it is largely used in the industrial applications. This method consists in describing the displacement of each substructure as a superposition both of the fixed-interface modes, obtained by clamping the boundaries, and of static deformations, defined from the interface DOF (Craig and Bampton, 1968).

The case of the clamped-clamped bar show in Fig. 1 is treated below as an example. By dividing the bar in two substructures, each one associate to one Young’s Modulus, we have for each substructure just one random variable, so that for $\rho = 1$ and 2:

$$
[K(E_\rho)] = [K_\rho]^{0} + \frac{1}{E_\rho^0} [K_\rho]^{0} \Delta E_\rho \quad (25)
$$

where $[K_\rho]$ is the stiffness matrix associated to each substructure ρ .

As shown in section 2 only the stiffness matrix have a Taylor expansion, and according to Craig and Bampton method, the Taylor's serie terms of the stiffness matrix of the assembled system are:

Zeroth-order:

$$[K]^0 = \begin{bmatrix} [K^R_{bb}]^0_1 + [K^R_{bb}]^0_2 & & \\ & [K^R_{m_1m_1}]^0 & \\ & & [K^R_{m_2m_2}]^0 \end{bmatrix} \tag{26}$$

First-order:

$$[K]^I_{E_1} = \frac{1}{E^0_1} \begin{bmatrix} [K^R_{bb}]^0_1 & & \\ & [K^R_{m_1m_1}]^0 & \\ & & [0_{m_2m_2}] \end{bmatrix} \tag{27}$$

$$[K]^I_{E_2} = \frac{1}{E^0_2} \begin{bmatrix} [K^R_{bb}]^0_2 & & \\ & [0_{m_1m_1}] & \\ & & [K^R_{m_2m_2}]^0 \end{bmatrix} \tag{28}$$

Second-order:

$$[K]^{II}_{E_\rho E_\sigma} = [0] \qquad \text{for: } \rho, \sigma = 1 \text{ and } 2 \tag{29}$$

Where $[K^R_{bb}]_s$ is the stiffness matrix reduced at the boundary of the substructure "s" and $[K^R_{m_sm_s}]$ is the constrained modes matrix of the substructure "s" reduced to the "m" first modes. The subscripts "b" are used to refer to the DOF number at the substructures interface ($b_1 = b_2 = b$) and "m_s" are used to refer to the retained modes number at the substructure "s".

The Taylor expansions terms involved into the equations (26) to (29) are:

a.) Stiffness Matrix reduced at the boundary of the substructure

For each substructure "s" ($s = 1$ and 2), we have:

Zeroth-order:

$$[K^R_{bb}]^0_s = [T]^T [K]^0_s [T] \qquad \text{with} \quad [T] = \begin{bmatrix} [I]_{bb} \\ [\Phi^*]_{ib} \end{bmatrix} \tag{30}$$

Where : $[I]_{bb}$ denotes the unit matrix with dimension "bb" associated to the boundary DOF and $[\Phi^*]_{ib}$ denotes the statics modes matrix, defined by:

$$[\Phi^*]_{ib} = -[K_{ii}]^{-1}[K_{ib}] \qquad \text{with} \quad i = \text{number of internals DOF at substructure} \tag{31}$$

First-order:

$$[K^R_{bb}]^I_{s_{E_\rho}} = [T]^T [K]^I_{s_{E_\rho}} [T] = \frac{1}{E^0_\rho} [T]^T [K]^0_s [T] \qquad \text{for : } \rho = 1 \text{ and } 2, \quad s = 1 \text{ and } 2 \tag{32}$$

then:

$$[K^R_{bb}]^I_{s_{E_\rho}} = \begin{cases} \frac{1}{E^0_\rho} [K^R_{bb}]^0_s & \text{for : } \rho = s, \\ 0 & \text{for : } \rho \neq s. \end{cases} \tag{33}$$

Second-order:

$$[K^R_{bb_s}]^{II}_{E_\rho E_\sigma} = [0] \qquad \text{for : } \rho, \sigma, s = 1 \text{ and } 2 \tag{34}$$

b.) Constrained modes matrix of the substructures reduced to the "m" first modes.

For each substructure "s", we have:

Zeroth-order:

$$[K^R_{mm_s}]^0 = [\Psi^*_s]^T [K_s]^0 [\Psi^*_s] \quad \text{and } s = 1 \text{ and } 2 \tag{35}$$

Where $[\Psi^*]$ is the constrained modes matrix, solution of the eigenproblem :

$$([K_{ii}]_s - \omega_j^2 [M_{ii}]_s)\{\psi^*_j\} = \{0\} \tag{36}$$

First-order:

$$[K^R_{m_s m_s}]^I_{E_\rho} = [\Psi^*_s]^T [K_s]^I_{E_\rho} [\Psi^*_s] = \frac{1}{E^0_\rho} [\Psi^*_s]^T [K_s]^0 [\Psi^*_s]$$
$$\text{for : } \rho = 1 \text{ and } 2, \quad s = 1 \text{ and } 2 \tag{37}$$

then:

$$[K^R_{m_s m_s}]^I_{E_\rho} = \begin{cases} \frac{1}{E^0_\rho} [K^R_{mm_s}]^0 & \text{for: } \rho = s, \\ \\ 0 & \text{for : } \rho \neq s. \end{cases} \tag{38}$$

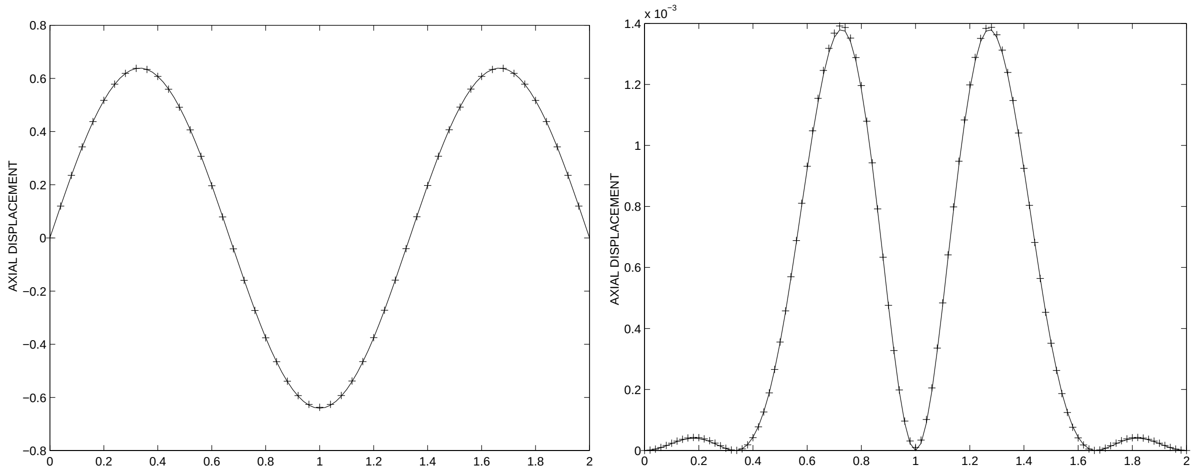
Second-order:

$$[K^R_{mm_s}]^{II}_{E_\rho E_\sigma} = [0] \qquad \text{for : } \rho, \sigma, s = 1 \text{ and } 2 \tag{39}$$

Substituting all those expressions into the eigenproblem of the assembled system we arrive to the equations similar to (5)-(7), but with a reduced DOF number. The mass matrix of the assembled system is not a function of the random variables and it is calculated using the classical equations of the Craig and Bampton method.

5. APPLICATION AND NUMERICAL RESULTS

Let us consider the clamped-clamped bar shown in Fig. 1, of length 2 $[m]$, diameter 0.02$[m]$ and density 7800 $[kg/m^3]$ with Young's moduli means $E_1 = E_2 = 21 \cdot 10^{10} [N/m^2]$. A Finite Element discretization is used, here with 100 elements. The comparison between the Perturbation method and the Monte Carlo simulation (30000 iterations) is shown in Fig. 2 for the third mode and a 5% standard deviation on the Young's moduli. The shape mode obtained by using Perturbation method is identical to Monte Carlo simulation. Even for larger standard deviation values, the results remain accurate.



(a) Mean value

(b) Standard deviation

Figure 2- Mean value and Standard deviation of the third eigenmode, for $\sigma_1 = \sigma_2 = 0.05$, Monte Carlo (+ +) and Perturbations method (—)

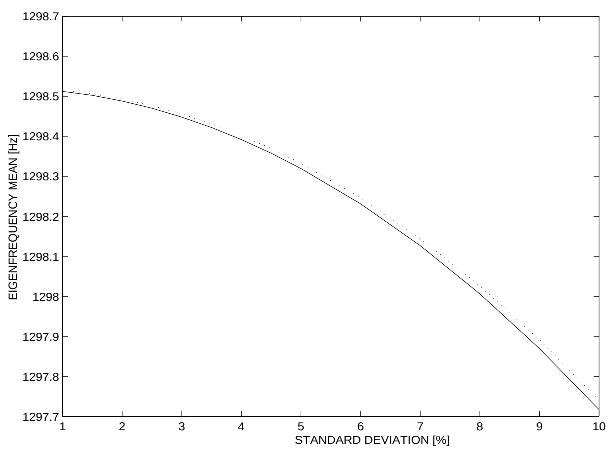


Figure 3- Evolution of the natural frequency mean versus the Young's modulus standard deviation values, Monte Carlo (—) and Perturbations method $(\cdots)$

The figure 3 shows the natural frequency evolution versus the Young's modulus standard deviation values. Both methods lead to identical values. However, the comparison between the FRF evaluated for both of the methods shows, for frequencies close to the natural frequencies of the system, a large difference between the results of the Perturbation method and Monte Carlo simulation. The figure 4 shows the mean values obtained by Perturbation method compared with Monte Carlo simulation.

Application of the modal synthesis is presented in the Fig. 5. The Monte Carlo simulation is computed for complete model and the 17 first modes are presented. The Perturbation solution is built by applying the stochastic Craig et Bampton method and we have used 5 eigenmodes for each substructure. As usual, substructuration yields truncated transfer function (the eigenmodes of higher frequencies are not taken into account).

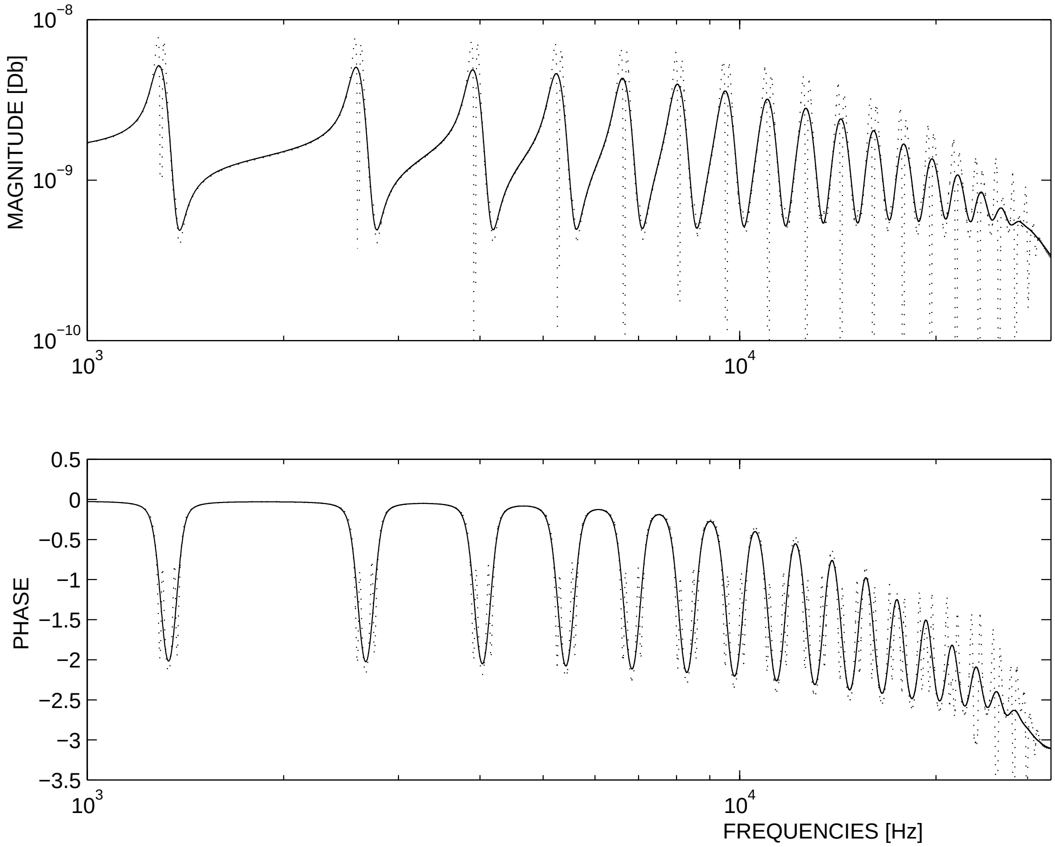


Figure 4- Mean Transfer function (magnitude and phase) for $\sigma_1 = \sigma_2 = 0.05$, Monte Carlo (—) and Perturbations method ($\cdots$).

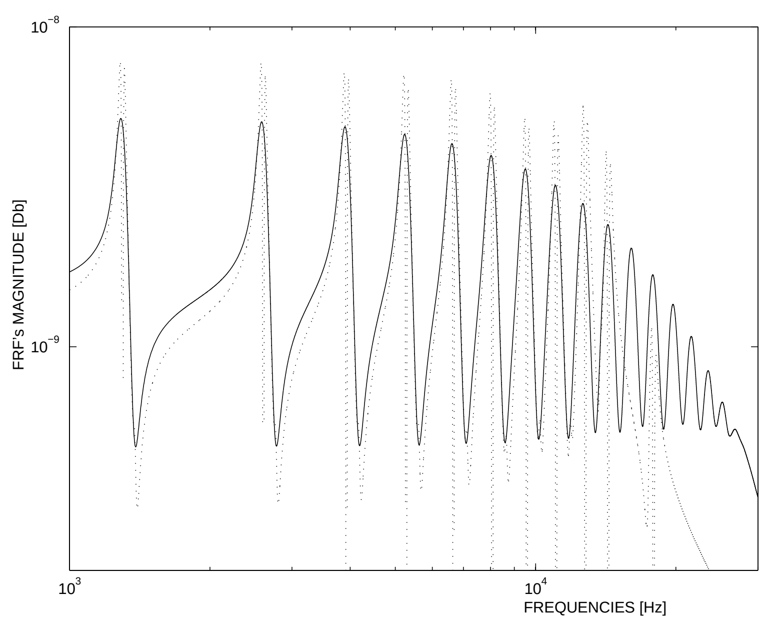


Figure 5- Mean Transfer function (magnitude) for $\sigma_1 = \sigma_2 = 0.05$, Monte Carlo (—) and Perturbations method ($\cdots$) with substructuration with 5 modes for each substructure.

6. CONCLUSION

A stochastic Component Modal Synthesis method, based an expansion in Taylor series about the mean values, has been presented in this paper. The stochastic finite element method have been tested on dynamical problems with good results for the computation of eigenmodes, even for uncertainties with large magnitude. The main difficulty in the perturbation method is the computation of the derivatives. Substructuration is very important in stochastic methods, based an expansion in Taylor series. First, it reduces the number of DOF. Moreover each uncertainty can be associated with a substructure. Then the computations are quite simplified and also faster. For frequencies close to the natural frequencies of the system, the Perturbation method is no more valid, and the expansion becomes divergent. A possibility for solve this problem is to use a method, based on a Karhunen-Loeve decomposition coupled with a polynomial chaos expansion and a Galerkin projection (Ghanem & Spanos,1991), what will be object of future research.

The Stochastic Craig and Bampton method presented in this paper allows the computation of the vibration frequency and modes shape for the structures with random parameters with precision and a small cost of calculation.

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