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COMPARISON OF ERROR FOR DIFFERENT APPROXIMATION FUNCTIONS IN THE DUAL RECIPROCITY METHOD

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Abstract *Since the publication of a book about the Dual Reciprocity Method in 1992 there has been much research into the development of approximation functions employed with the method. Many new functions have been proposed and used to solve engineering problems. Given this wealth of new functions, it is important to be able to select an adequate function for a given problem, and in order to do this it is necessary to know something about the properties and behaviour of each. To this end in this paper several examples are solved using a wide range of functions and the results obtained with each function compared with a view to establishing tentative criteria for choosing the function.*

Key Words: *Dual Reciprocity; Boundary Elements; Approximation Functions; Comparative Error*

1. INTRODUCTION

The Dual Reciprocity Method is a means of dealing with equations in BEM for which a complete fundamental solution is either unavailable or inconvenient, leading to one or more terms being "left over" as domain integrals after the usual operations on the Boundary Integral Equation. The method permits these terms to be expressed as integrals over the boundary in a systematic way. Using DRM we can apply Boundary Elements to a great many problems in engineering with a knowledge of only a few basic fundamental solutions. All additional terms (non linear effects, time derivatives etc) are treated as body forces and taken to the boundary using the method. The method was first introduced by Nardini and Brebbia, (1982). The method has a disadvantage of introducing an additional approximation, the DRM approximating functions, f to be discussed here. In the original work, the function $f = 1 + r$ was employed, being adopted by most other researchers until the beginning of this decade. With the publication of a book collecting applications to that point (Partridge et al, 1992), the number of researchers interested in the subject increased considerably. The function r was identified as being a "radial basis function" or RBF, well known to mathematicians but new to engineers. From then on a relatively large number of functions have been used successfully and many of them have been claimed to be a good or even "optimal" choice by at least one author, thus creating a potentially confusing situation for those interested in simply using the method.

At the current state of the art it is not possible to define one single function and claim that this is the best for all problems. A more practical approach to the question of selecting a function is to establish which are appropriate for different classes of problems and what are their limitations. In this context, in this paper, different types of problem are considered, and results presented using a range of functions in an attempt to bring out their advantages and disadvantages and to suggest which functions are appropriate for each type of application.

The examples considered are limited to potential type problems in two dimensions, the three dimensional case requires a separate study; an appropriate function for a 3D problem is not necessarily the equivalent of that for the 2D case.

2. DRM APPROXIMATING FUNCTIONS

The original DRM Approximation Function r was used by Nardini and Brebbia, combined with the constant 1. r^3 was found to be an improvement on r by Yamada et al, (1994) and Zhang & Zhu (1994). The work by Yamada et al, (1994) followed mathematical work done by Powell, (1990) and both of the functions r and r^3 were identified as belonging to a class of functions known as Radial Basis Functions or RBF. The even powers of r (eg r^2) are not RBFs and, if used on their own in DRM, produce rubbish solutions. If combined with other legitimate RBFs, they usually effect very little the solution, though it is better to avoid them completely. Another RBF, the Thin Plate Spline or TPS $r^2\log(r)$ has been employed for problems involving body forces of high order, (Karur & Ramachandran, 1994). More recently the Higher Order Thin Plate Splines, for instance $r^4\log(r)$ are being mentioned. Other RBFs cited in connection with DRM involve the use of user defined constants, for instance the Multiquadric and Gaussian RBFs, (Karur & Ramachandran, 1994). The RBFs are also known as "local " functions given that they interpolate only in the neighbourhood of a given node. The mathematician Golberg, (1995), who has made several important contributions to the method, has manifested his surprise at the intuitive choice of the function r by the authors of the original paper, Nardini & Brebbia, (1982). The present author feels that this function was probably used because it was the only one on the list in the original paper that could have been made to work at the time. In 1982 engineers were unaware of RBFs or the software capable of inverting the nearly singular matrices associated with the remaining (global) functions, see below. The surprising thing is not that r was chosen, but that it was included in the list, if that function had not been "hit upon" DRM might never have existed. For more than 10 years it was sufficient that *a* function existed which made the method operable.

The global approximating functions mentioned by Nardini and Brebbia, (1982), but not employed by them are, for example the terms from the Pascal Triangle, 1, x, y etc, used to develop finite element shape functions, the sine expansion $\sin(x)$, $\sin(y)$, and other sets of functions. These functions have been investigated for use in DRM by Cheng et al, (1993) who give details about how to implement them computationally.

The ATPS (Augmented Thin Plate Spline) was presented after a theoretical mathematical study by Golberg & Chen (1994) as to what might be the optimum function for DRM, and has been used successfully for many problems for instance in Bridges & Wrobel, (1996). This function consists of the TPS combined with the augmentation functions 1, x, y, in such a way that the function is neither local or global, but a combination of the two. Details of the implementation of this function are given in Bridges & Wrobel (1996). However, in a later paper Golberg revised his position, presenting the Augmented Multiquadric as the optimal interpolation function, (Golberg et al, 1996)

Given the nature of the ATPS, which is an RBF combined with three terms from a global expansion, the present author postulated the generalization of this concept to the Hybrid Functions, (Partridge & Sensale, 1997) in which an RBF is combined or augmented with terms from a global expansion defined by the order and nature of the body force term to be approximated. Examples of such functions are SAPT3 and TAGS4. SAPT3 consists of a R^3 function Augmented with up to the 3rd line of terms from the Pascal Triangle, ie terms 1, x, y, x^2 xy and y^2. TAGS4 consists of a TPS augmented with the 9 terms $\sin(x)$, $\sin(y)$, ...$\sin(3y)$. The number at the end of each Hybrid function name is the number of lines from the Pascal Triangle to be included, and is thus one higher than the order of augmentation. Hybrid Functions can be "designed" for a specific problem if necessary. One proviso: only *Complete Sets* of such functions should be employed, (Cheng et al, 1993).

Thus one can employ local functions, Global functions or those which combine the two types. The use of Global functions on their own requires special procedures as the matrices of coefficients **F** tends to be nearly singular. Some comparisons with early local functions can be found in (Partridge, 1995). This author feels these functions are much better employed combined with the RBFs in the third category than on their own. The mechanism of the combination, (Bridges & Wrobel, 1996) avoids completely the problems of ill conditioning of the matrices that are found if these functions are used alone. Thus the global functions on their own will not further be considered here.

The RBFs with user defined constants, particularly the multiquadric, is popular amongst researchers who use these functions in methods other than DRM. The use of these functions in DRM transfers the discussion from which is the most appropriate function to which is the best value of the constant, introducing an additional optimization problem. A procedure for executing this optimization is given in (Golberg et al, 1996). The parameter to be optimized, c in $(r^2+c^2)^{1/2}$, is mesh dependent, One value for the whole problem can be used for small problems, for problems involving subregions a different value is required in each. Values for each point used can be calculated if desired.

In the case of problems involving derivatives, the function r and the TPS will contain an indeterminacy at $r = 0$ in $\frac{\partial f}{\partial x}$. This does not prevent these functions from being used if the indeterminate value is fixed to zero. Further differentiation will produce a singularity at these points, making the use of these functions inviable. r^3 and $r^4 \log r$ do not contain this indeterminacy in $\frac{\partial f}{\partial x}$. Of the functions in current use only the multiquadric may be differentiated indefinitely without producing indeterminacies or singularities. The same seems to be true of the Gaussian function, but this is not in current use in DRM, perhaps due to difficulties in finding an appropriate particular solution.

The Dual Reciprocity Method itself is well known, and will not be described here, a full description is given in Partridge et al, (1992).

3. NUMERICAL EXAMPLES

3.1 Considerations about the presentation of error

Many different schemes can be found in the literature for presenting the error encountered in numerical solutions. The most common are to express the same as a mean percentage, ϵ_m, the use of the difference in nodal values, ϵ_d, or the RMS error, ϵ_r :-

$$\epsilon_m = \frac{1}{n} \sum \left(\frac{u_{\text{calculated}} - u_{\text{exact}}}{u_{\text{exact}}} \right) \times 100$$

$$\epsilon_d = u_{\text{exact}} - u_{\text{calculated}}$$

$$\epsilon_r = \sqrt{(\sum(u_{\text{exact}} - u_{\text{calculated}})^2/n} \quad (1)$$

In the above, u is the problem variable and n is the number of points considered.

ϵ_m can overestimate error, a small value of u_{exact} can contribute over proportionately to the result. Points with zero values of the exact solution cannot be taken into consideration, however accurately or inaccurately the solution is calculated. In addition, the question of "average" values can also conceal large differences in quality of results. ϵ_d can underestimate error, numbers of very small order being common. The best way of using this is to present the largest error encountered. The RMS error, ϵ_r avoids the problems mentioned above and will be used here unless stated otherwise.

ϵ_r appears to have the following advantage: the number of zeros after the decimal allows one to infer the number of decimal places calculated correctly. Some examples: $\epsilon_r = 0.0008$: in this case one can only expect two decimal places of accuracy in the numerical results, though some will be correct to three places. $\epsilon_r = 0.0001$: here nearly all results will be correct to three decimals.

3.2 Problem 1: known function as a body force

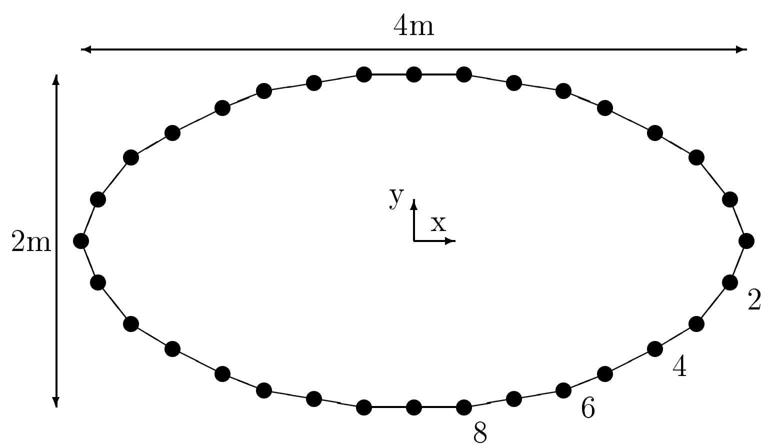

Fig. 1 Elliptic Geometry

Here the problem considered is that of eq. (2)

$$\nabla^2 u = -x^2 \quad (2)$$

on an elliptical geometry with a semi-major axis of length 2m and a semi minor axis of length 1m and the origin of coordinates in the center as shown in fig. 1, and subject to a boundary condition $u = 0$ on Γ. Quadratic Elements are used. It can be shown that the exact solution for u is given by

$$u = (50x^2 - 8y^2 + 33.6)(x^2 + y^2 - 1)/246 \quad (3)$$

from which q on Γ can easily be obtained

$$q = -[(50x^3 + 96y^2x - 83.2x)n_1 + (-32y^3 + 96yx^2 + 83.2)n_2]/246 \quad (4)$$

where $n_1 = x/R$ and $n_2 = 4y/R$ for $R = \sqrt{(x^2 + 16y^2)}$. Please note that the exact solutions for q for this and similar problems on elliptical geometries given in Partridge et al, (1992) should be

Table 1: Results for problem 1

	Discretization #1 16 Elements; 7 Internal Nodes						
	r	r^3	TPS	ATPS	SAPT2	SAPT3	Exact
2	-1.005	-0.980	-0.971	-1.013	-0.996	-0.985	-0.984
4	-0.860	-0.848	-0.834	-0.862	-0.856	-0.854	-0.856
6	-0.478	-0.490	-0.465	-0.489	-0.496	-0.502	-0.502
8	-0.202	-0.227	-0.193	-0.223	-0.235	-0.244	-0.244
ϵ_r in q	0.026	0.019	0.035	0.028	0.022	0.018	
ϵ_r in u	0.0077	0.0033	0.011	0.0036	0.0012	0.00019	
	Discretization #2 16 Elements; 17 Internal Nodes						
	r	r^3	TPS	ATPS	SAPT2	SAPT3	Exact
2	-0.995	-0.983	-0.980	-0.999	-0.991	-0.985	-0.984
4	-0.860	-0.852	-0.847	-0.858	-0.855	-0.854	-0.856
6	-0.494	-0.498	-0.491	-0.500	-0.501	-0.502	-0.502
8	-0.229	-0.240	-0.231	-0.239	-0.242	-0.244	-0.244
ϵ_r in q	0.010	0.016	0.016	0.023	0.022	0.018	
ϵ_r in u	0.0023	0.00046	0.0019	0.00077	0.00022	0.00013	
	Discretization #3 32 Elements; 17 Internal Nodes						
	r	r^3	TPS	ATPS	SAPT2	SAPT3	Exact
2	-0.999	-0.985	-0.982	-1.001	-0.992	-0.987	-0.984
4	-0.860	-0.853	-0.849	-0.860	-0.857	-0.856	-0.856
6	-0.492	-0.498	-0.491	-0.499	-0.501	-0.502	-0.502
8	-0.227	-0.239	-0.231	-0.238	-0.242	-0.244	-0.244
ϵ_r in q	0.012	0.0036	0.0094	0.0076	0.0042	0.0025	
ϵ_r in u	0.0026	0.00050	0.0019	0.00080	0.00018	0.00003	
	Discretization #4 64 Elements; 17 Internal Nodes						
	r	r^3	TPS	ATPS	SAPT2	SAPT3	Exact
2	-1.000	-0.982	-0.973	-0.996	-0.987	-0.984	-0.984
4	-0.860	-0.853	-0.849	-0.859	-0.856	-0.856	-0.856
6	-0.492	-0.498	-0.491	-0.499	-0.501	-0.502	-0.502
8	-0.226	-0.240	-0.231	-0.238	-0.242	-0.244	-0.244
ϵ_r in q	0.013	0.0030	0.0092	0.0066	0.0018	0.00020	
ϵ_r in u	0.0026	0.00052	0.00092	0.00079	0.00017	0.000001	

corrected, (Santiago Cruz, 1996). For the analyses carried out here, an LU decomposition solver with partial pivoting was used and singular integration was carried out using a special logarithmic numerical integration table.

Results are given in table 1 for a selection of approximating functions, for four different discretizations, the number of boundary elements and internal nodes in each being as indicated. Nodal results for q on the boundary are given for the four nodes numbered in fig. 1. After these results, for each discretization, the RMS error ϵ_r for q on the boundary and for u at internal nodes is given for that discretization for each f considered.

Overall, r^3 does better than r, though the use or the RMS as an error measure diminishes considerably the perceived advantage of the former. The relation can be expressed as follows: results for most nodes are much better with r^3, however at some points there is a bigger error with this function, detected better with RMS. r improves better with increased internal points, r^3 improves with more boundary elements. Neither TPS or ATPS behaves better than r^3, the latter has only linear augmentation functions whereas the problem has a quadratic body force. If mean percentage error, ϵ_m, is used, ATPS comes out better than r^3. Best results are produced by the SAPT3 Hybrid function with quadratic augmentation. Results for TAPT3 and RAPT3 (omitted) are the same as those for SAPT3. When augmentation functions which include the body force term, in this case x^2 in eq. (2), are used, boundary results become independent of the RBF basis function employed, (Partridge & Sensale, 1997), and also of number of internal nodes: SAPT3 repeats boundary results for meshes #1 and #2. Note that SAPT2 does better than ATPS. (SAPT2 uses the r^3 basis function whereas ATPS uses the TPS: both have linear augmentation).

If one defines an acceptable error for this problem as $\epsilon_r \leq 0.0002$, that is nearly all results will be correct to 3 decimals, acceptable internal results are given by SAPT3 on mesh #1 and #2, and by SAPT2 and SAPT3 on meshes #3 and #4. Internal results for r^3 and ATPS are close on meshes #2, #3 and #4. All results using r, and all results using the TPS without augmentation except for mesh #4, are inaccurate. In the case of boundary results, acceptable accuracy is reached on mesh #4 by SAPT3. Thus boundary results converge more slowly than results for u at interior nodes, the former require a much finer discretization than the latter in order to obtain good accuracy. It is surprising to note that a greater accuracy on the boundary can be obtained in elasticity problems with known body forces than is the case here for potential problems, (Partridge & Sensale, 1997).

The Multiquadric function has been used to solve the problem under consideration in Golberg et al, (1996), in which an optimization analysis is carried out first to fix the value of the shape parameter. Results are given for a discretization involving 16 boundary nodes and 17 internal points for u in the interior, calculated to an accuracy of $0.0000007 \leq \epsilon_d \leq 0.0005$. This level of accuracy is compatible with that obtained by SAPT3 on #1 and SAPT2 and SAPT3 on meshes #2, #3 and #4. The method used in Golberg et al, (1996), which differs from "traditional" DRM, thus needs less nodes to obtain accurate results, however the DRM solutions presented here do not require optimization, and in the case of SAPT2 and SAPT3 convergence is obtained by increasing the number of boundary elements.

Thus problems with known functions as body force terms can be solved accurately in at least three ways: by using the Multiquadric, by using a appropriate Hybrid function in DRM or by the well known Method of Particular Solutions.

3.3 Problem 2: unknown function as body force

In this case the body force term involves the unknown function u, however no derivatives are present. The problem considered is that of eq. (5)

$$\nabla^2 u = -u \tag{5}$$

The same geometry as in the previous problem is considered. A boundary condition $u = \sin(x)$ is imposed on Γ in such a way that

$$q = -\cos(x) n_1 \tag{6}$$

where n_1 is as in eq. (4). The same discretizations are considered as in the previous example. Results are given for RMS error, ϵ_r in table 2 for u and q for each f.

In order to obtain multiquadric results, this function was incorporated into a standard DRM code. The algorithm for obtaining the shape parameter described in Golberg et al, (1996), does not apply in this case as the body force $-u$ in eq. (5) is an unknown function. To circumvent this, and given the range of values of c given for the previous example in Golberg et al (1996), a search was carried out for each mesh starting from $c = 0.5$ using increments of 0.5: on finding a minimum, a local search with increments of 0.1 was carried out: the c used was that which produced the lowest ϵ_r for q on the boundary. This procedure is not suitable for general use and is only introduced here in order to obtain values for comparison. The shape parameter cannot be fixed arbitrarily as values outside a small range will produce increasing error and eventual rubbish solutions. The values encountered were: mesh #1 $c = 1.5$, mesh #2 $c = 0.8$, mesh #3 $c = 1.2$, mesh #4 $c = 0.6$.

Table 2: RMS Error for results for problem 2

	Discretization #1; 16 Elements, 7 Internal Nodes						
ϵ_r	r	r^3	$r^4 \log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.0044	0.0045	0.0050	0.0043	0.0049	0.0049	0.0033
u	0.0013	0.00019	0.00062	0.0002	0.00011	0.00011	0.00013
	Discretization #2; 16 Elements; 17 Internal Nodes						
	r	r^3	$r^4 \log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.0031	0.0044	0.0039	0.0039	0.0049	0.0049	0.0030
u	0.00052	0.000056	0.00010	0.000057	0.000040	0.000040	0.000047
	Discretization #3; 32 Elements, 17 Internal Nodes						
	r	r^3	$r^4 \log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.0023	0.0015	0.0013	0.0015	0.0012	0.0012	0.00088
u	0.00047	0.000059	0.00010	0.000066	0.000020	0.000020	0.000033
	Discretization #4; 64 Elements, 17 Internal Nodes						
	r	r^3	$r^4 \log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.0022	0.00074	0.0010	0.00081	0.00032	0.00032	0.0011
u	0.00046	0.000060	0.00011	0.000067	0.000020	0.000020	0.000077

To fix an acceptable error for this case it must be remembered that one cannot expect the same level of accuracy in problems for which the body force is an unknown function. Results show about 3 decimal places of accuracy for q on the boundary in the case of the Multiquadric on mesh #3, and r^3, SAPT2 and VAPT2 on mesh #4. Results for u on the interior are an order of magnitude more accurate, three decimals of accuracy being obtained for all of the functions considered from mesh #2 onwards. The lowest error is shown by the Hybrid function VAPT2 (a higher order TPS with linear augmentation), $\epsilon_r = 0.00002$ on mesh #4. VAPT3 with quadratic augmentation repeats VAPT2. The multiquadric diverges on mesh #4, this function does not need such a high density of boundary nodes, and produces accurate results on mesh #3. r^3, SAPT2, VAPT2 and the multiquadric perform best for this case.

3.4 Problem 3: unknown function including derivatives as body force

Here the problem considered is that of eq. (7)

$$\nabla^2 u = -\frac{\partial u}{\partial x} \tag{7}$$

The same geometry and discretizations used in the previous problem are considered. A boundary condition $u = \exp(-x)$ is imposed on Γ in such a way that

$$q = -\exp(-x)n_1 \tag{8}$$

where n_1 is as in eq. (4). First, results for RMS error will be given for the meshes mentioned above in table 3.

Table 3: RMS Error for results for problem 3

	Discretization #1; 16 Elements, 7 Internal Nodes						
ϵ_r	r	r^3	$r^4\log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.17	0.054	0.060	0.058	0.055	0.054	0.057
u	0.030	0.0046	0.0027	0.0055	0.0026	0.0018	0.0050
	Discretization #2; 16 Elements, 17 Internal Nodes						
	r	r^3	$r^4\log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.11	0.054	0.055	0.052	0.054	0.054	0.055
u	0.011	0.00073	0.0010	0.0017	0.00058	0.00063	0.0025
	Discretization #3; 32 Elements, 17 Internal Nodes						
	r	r^3	$r^4\log r$	SAPT2	VAPT2	VAPT3	$(r^2+c^2)^{1/2}$
q	0.11	0.013	0.013	0.019	0.011	0.011	0.015
u	0.011	0.00071	0.0010	0.0017	0.00056	0.00062	0.0014

It will be immediately noticed that there has been a reduction of accuracy of an order of magnitude in relation to the results for the previous problem. Results for q on the boundary are not acceptable for any of the f, results on the interior are acceptable for r^3 and the VAPT functions on mesh #3. The Multiquadric does not produce the same accuracy as in the previous two problems. However in the analysis of convective type problems, one is not normally interested in q on the boundary, it is the u result that is of more importance. Reasonable results for u at internal points are produces by r^3 and the VAPT Hybrid functions on meshes #2 and #3.

3.5 Problem 4: a non linear case

Here the problem considered is that of eq. (9)

$$\nabla^2 u = -u\frac{\partial u}{\partial x} \tag{9}$$

The same geometry as in problems 1-3 is considered. A boundary condition $u = \frac{2}{x}$ is imposed on Γ which is a particular solution to eq. (9) from which q may easily be calculated. The origin is moved to the point (-4,0) in fig. 1 to avoid the singularity at $x = 0$. The same discretizations are considered as in previous examples on this geometry. RMS errors for q and u are given in table 4 for the meshes considered. Convergence was obtained in 5 iterations.

Table 4: RMS Error for results for Problem 4

	Discretization #1; 16 Elements, 7 Internal Nodes					
ϵ_r	r	r^3	$r^4\log r$	SAPT2	VAPT2	VAPT3
q	0.0069	0.0048	0.0088	0.0046	0.0044	0.0040
u	0.0015	0.00019	0.00061	0.00024	0.00019	0.00011
	Discretization #2; 16 Elements, 17 Internal Nodes					
	r	r^3	$r^4\log r$	SAPT2	VAPT2	VAPT3
q	0.0053	0.0039	0.0048	0.0048	0.0049	0.0039
u	0.00050	0.00014	0.00026	0.00010	0.00006	0.00005
	Discretization #3; 32 Elements, 17 Internal Nodes					
	r	r^3	$r^4\log r$	SAPT2	VAPT2	VAPT3
q	0.0045	0.0016	0.0027	0.0011	0.0012	0.00082
u	0.00052	0.00014	0.00026	0.00010	0.00004	0.00004

Considering table 4 it may be noticed that the accuracy for the non-linear problem is similar to that of the problem without derivatives (problem 2), probably due to the iterative process used to obtain the solution. The smallest error in the case of all three meshes is that of the Hybrid function, VAPT3. Also, excluding r, the difference between the error results for the different f functions is not that great.

CONCLUSIONS

In considering which f function should be used for a given problem the most important factor to take into account is the nature of the domain integral to be taken to the boundary:

In the case of known functions the problem can be accurately solved using the Multiquadric, or an appropriate Hybrid function, in the latter case the results converge as the mesh is refined, and for augmentation which includes the body force term, internal nodes are necessary only to guarantee the invertability of the **F** matrix. This type of problem may also be solved accurately using the Particular Solution method. In the case of the Multiquadric, a prior optimization study is necessary to determine the best value of the shape parameter.

In the case of unknown functions without derivatives the multiquadric can be used if a value of the shape parameter can be determined, the optimization cannot be carried out in the usual way as the domain integral involves an unknown function. Otherwise linear augmented r^3 and $r^4\log r$ are good options.

In the case of unknown functions with derivatives the choice for f falls on linear augmented r^3 and $r^4\log r$. r and the TPS should be avoided and VAPT3 may be considered. A high density of internal nodes is advisable in this case to guarranttee a proper description of the body force derivative over the domain.

In the case of non-linear functions once again, the choice for f falls on linear augmented r^3 and $r^4\log r$. Once again r and the TPS should be avoided and VAPT3 may be considered.

The results presented here and other results obtained by the author but not included indicate that the TPS function, in its low order version $r^2\log r$ is inferior in performance to r^3, and is overrated in the literature, and that the higher order Thin Plate Splines, for example $r^4\log r$ and associated Hybrid functions are worthy of further study.

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