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STIFFENING EFFECTS ON THE FREE VIBRATION BEHAVIOR OF COMPOSITE PLATES WITH PZT ACTUATORS

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Abstract. *The free vibration behavior of thin composite plates with surface bonded piezoelectric patches including stress stiffening effects is investigated. A finite element formulation is presented, based on the Reissner-Mindlin theory and including nonlinear strain-displacement relations, to formulate a free vibration eigenvalue problem with the presence of a geometrical stiffness matrix. The case of symmetric laminates and ideal linear behavior is assumed for the piezoelectric actuation. Due to the absence of membrane-bending coupling, in-phase applied voltages produce only inplane induced piezoelectric stress resultants, which are assumed proportional to the applied voltages. Examples with numerical results for unconstrained plates equipped with piezoelectric actuators show that inplane induced stresses may significantly affect the free vibration behavior.*

Key Words*: Piezoelectric, plate finite element, stress stiffening*

1. INTRODUCTION

Piezoelectric materials can be used in smart structures as sensors or actuators in applications such as shape control, active damping and acoustic noise suppression. The design of such systems requires accurate electromechanical models to simulate the interaction between the structure and the piezoelectric elements. The recent literature addresses the modeling of piezoelectric elements either bonded or embedded to several different types of structures. The piezoelectric actuation of beams was treated in depth by Crawley and co-workers (Crawley & Anderson, 1990; Crawley & de Luis, 1987). Several other formulations were also presented for the modeling of plates (Ha et al., 1992; Batra et al., 1996; Ghosh & Batra, 1995), cylindrical shells (Sonti & Jones, 1996), and general shells (Tzou & Yeh, 1996).

In composite laminates piezoelectric elements usually consist of patches symmetrically bonded to the top and bottom surfaces. The voltages applied to the piezoelectric elements may induce inplane, bending and localized shear deformations in the laminate (Chandra & Chopra, 1993). For symmetric laminates, applying the same in-phase voltage to both patches will cause only inplane deformation, whereas equal out of phase voltage will cause pure bending. Combined inplane and bending deformation may be induced when different voltages are applied to each patch or when one of the elements is used as actuator and the other as sensor. The same situation occurs if a single piezoelectric actuator is mounted on a surface.

The presence of inplane deformation may have a significant influence on the mechanical behavior of plates, affecting the flexural stiffness and so the dynamic and stability characteristics of isotropic (Brunelle & Robertson, 1974) and laminated plates (Yang & Shieh, 1987). Rammerstorfer (1977) determined optimum fields of residual inplane stresses that maximize the first natural frequency and buckling load of plates. Almeida and Hansen (1997) showed that, with proper design, inplane thermal residual stresses from the curing process can be tailored to significantly enhance the mechanical behavior by increasing the critical buckling load of symmetric composite plates.

Piezoelectric actuators can be used with great versatility to induce favorable inplane stresses with the purpose of improving the mechanical behavior of composite structures, since the magnitude of the inplane induced stress can be effectively controlled by varying the voltage applied to each actuator. The consideration of the stress stiffening effects on laminate behavior requires the inclusion of geometric stiffness in the structural analysis.

The purpose of this work is to investigate the problem of free vibration behavior of composite plates equipped with surface bonded piezoelectric elements, considering the stress stiffening effect caused by the inplane induced deformation. With this in mind, nonlinear strain-displacement relations are used to formulate a finite element model including the geometric stiffness associated with the inplane piezoelectric induced stress. Examples illustrate how induced stress stiffening influences the free vibration behavior of unconstrained rectangular plates equipped with some suggestive configurations of piezoelectric patches.

2. PROBLEM FORMULATION

The free vibration analysis for laminated plates based on the Reissner--Mindlin theory, including stress stiffening effects from the piezoelectric actuation is formulated. The solution is subdivided in two parts. Firstly a linear static analysis of the structure subjected to induced inplane piezoelectric stresses is performed, from where the stresses over the plate are determined. After the stresses are known, a free-vibration analysis is carried out, however considering the effects of the induced piezoelectric stress field in the geometric stiffness matrix.

2.1 Induced inplane stresses problem

Consider a composite laminated plate where each lamina is orthotropic and oriented at an angle θ with respect to the y axis. According to Mindlin plate hypothesis, the inplane displacements $\bar{u}$ and $\bar{v}$ vary linearly through the plate thickness and the transverse displacement $\bar{w}$ is assumed constant through the plate thickness. Throughout this work, an overline indicates a quantity at an arbitrary point *(x,y,z)* in the plate and quantities without the overline are defined on the *x-y* plane at the plate midsurface.

The following vectors of midsurface inplane strains, curvatures and out of plane shear deformations, are defined respectively as

$$
\begin{aligned}
\{\varepsilon_0\}^T &= \begin{bmatrix} u_{,x} & v_{,y} & u_{,y}+v_{,x} \end{bmatrix} \\
\{\kappa\}^T &= \begin{bmatrix} \psi_{x,x} & \psi_{y,y} & \psi_{x,y}+\psi_{y,x} \end{bmatrix} \\
\{\Gamma\}^T &= \begin{bmatrix} w_{,x}+\psi_x & w_{,y}+\psi_y \end{bmatrix}
\end{aligned}
\tag{1}
$$

where u, v, and w are the midsurface displacements along directions x, y, and z, respectively, and $\psi_{,x}$ and $\psi_{,y}$ are rotations of the normal to the undeformed midsurface in the x-z and y-z planes respectively.

The constitutive relation including the piezoelectric material is assumed linear. It is also assumed that the piezoelectric elements are thin and the electric field is parallel to z, that is, $\{E\}^T = [\ 0 \quad 0 \quad E_3]$. Moreover, it is assumed that E_3 is constant within each piezoelectric element and is given approximately by $E_3 = V / t_p$, where V is the voltage at the piezoelectric element and t_p is its thickness. Hence, the constitutive equations for stresses becomes:

$$\begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\sigma}_{xy} \\ \bar{\sigma}_{xz} \\ \bar{\sigma}_{yz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{xz} \\ \bar{\gamma}_{yz} \end{Bmatrix} - E_3 \begin{Bmatrix} e_{31} \\ e_{31} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

where $\{\bar{\sigma}\}$ are the stresses; $\{\bar{\varepsilon}\}$ are the strains; $[\bar{Q}]$ is the elasticity matrix; $e_{31}=e_{32}$ are the piezoelectric constants; and E_3 is the electric field vector.

We now introduce the following matrices,

$$\begin{aligned} \left(A_{ij}, B_{ij}, D_{ij}\right) &= \int_{-h/2-t_p}^{h/2+t_p} \left(1, z, z^2\right) \bar{Q}_{ij}\, dz \qquad i, j = 1, 2, 6 \\ A_{kl}^* &= \int_{-h/2-t_p}^{h/2+t_p} \bar{Q}_{kl}\, dz \qquad\qquad\qquad k, l = 4, 5 \end{aligned} \tag{3}$$

and the piezoelectric force and moment stress resultants along the thickness are respectively:

$$\left(\{N_p\}, \{M_p\}\right) = \int_{-h/2-t_p}^{h/2+t_p} (1, z) E_3 \{e_p\}\, dz \tag{4}$$

where, $\{e_p\}^{\mathrm{T}} = [\ e_{31} \quad e_{31} \quad 0\ ]$ is the vector of piezoelectric constants. With these definitions the potential energy becomes,

$$\pi = \frac{1}{2} \int_A \begin{Bmatrix} \{\varepsilon_0\} \\ \{\kappa\} \\ \{\Gamma\} \end{Bmatrix}^T \begin{bmatrix} [A] & [B] & [0] \\ [B] & [D] & [0] \\ [0] & [0] & [A^*] \end{bmatrix} \begin{Bmatrix} \{\varepsilon_0\} \\ \{\kappa\} \\ \{\Gamma\} \end{Bmatrix} dA - \int_A \begin{Bmatrix} \{\varepsilon_0\} \\ \{\kappa\} \\ \{\Gamma\} \end{Bmatrix}^T \begin{Bmatrix} \{N_p\} \\ \{M_p\} \\ \{0\} \end{Bmatrix} dA \tag{5}$$

For the sake of completeness the equations presented in this section include bending-membrane coupling terms. However, due to the symmetry assumed with respect to the z axis, the bending-membrane coupling matrix $[B]$, as well as the vector of piezoelectric moment stress resultant along the thickness, $\{M_p\}$, will be identically zero.

2.2 Free vibration problem

The next step in the formulation consists of a free vibration problem for a composite plate subject to initial piezoelectrically induced stresses. The free vibration equations of motion for this problem can be derived from Hamilton's principle,

$$\int_{t_1}^{t_2} \delta\left(T - U - \Delta W\right) dt = 0 \tag{6}$$

where T is the kinetic energy, U is the strain energy and ΔW is the change in the potential energy due to the work of inplane stresses. The inplane elastic stresses due to piezoelectric actuation, can be made available from the static stress analysis described in the last section, Eqn. (2). Assuming symmetric laminates, the following inplane piezoelectric stress resultants are defined:

$$\begin{aligned} \{N^0\} &= \int_{-h/2-t_p}^{h/2+t_p} \{\bar{\sigma}^0\}\, dz = [A]\{\varepsilon_0\} - \{N_p\} \\ \{L^0\} &= \int_{-h/2-t_p}^{h/2+t_p} \{\bar{\sigma}^0\}\, z^2 dz = [D]\{\varepsilon_0\} - \{P_p\} \end{aligned} \tag{7}$$

Notice that the terms L_{ij} depending on the initial stresses are not zero even for the case of symmetric laminates.

The work of these piezoelectric stresses through the plate shortening caused by bending deflection leads to the following change in the bending potential energy (Dawe & Roufaeil, 1982):

$$\Delta W = -\int_V \left(\bar{\sigma}_x^0\, \bar{\varepsilon}_x^N + \bar{\sigma}_y^0\, \bar{\varepsilon}_y^N + \bar{\sigma}_{xy}^0\, \bar{\gamma}_{xy}^N\right) dx\, dy\, dz \tag{8}$$

where $\{\bar{\varepsilon}^N\}$ are the nonlinear or second order strain components. Considering only the terms related to bending displacements, i.e., zeroing the midsurface inplane displacements, leads to the following nonlinear strain components,

$$\begin{aligned} \bar{\varepsilon}_x^N &= \frac{1}{2}\left[z^2\left(\psi_{x,x}^2 + \psi_{y,x}^2\right) + w_{,x}^2 \right] \\ \bar{\varepsilon}_y^N &= \frac{1}{2}\left[z^2\left(\psi_{x,y}^2 + \psi_{y,y}^2\right) + w_{,y}^2 \right] \\ \bar{\gamma}_{xy}^N &= w_{,x} w_{,y} + z^2\left(\psi_{x,x}\psi_{x,y} + \psi_{y,x}\psi_{y,y}\right) \end{aligned} \tag{9}$$

Using Eqns. (7-9), the change of potential energy then becomes

$$\Delta W = -\frac{1}{2}\int_A \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix}^T \begin{bmatrix} N_x^0 & N_{xy}^0 \\ N_{xy}^0 & N_y^0 \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} dx\, dy - $$
$$-\frac{1}{2}\int_A \begin{Bmatrix} \psi_{x,x} \\ \psi_{x,y} \\ \psi_{y,x} \\ \psi_{y,y} \end{Bmatrix}^T \begin{bmatrix} L_x^0 & L_{xy}^0 & 0 & 0 \\ L_{xy}^0 & L_y^0 & 0 & 0 \\ 0 & 0 & L_x^0 & L_{xy}^0 \\ 0 & 0 & L_{xy}^0 & L_y^0 \end{bmatrix} \begin{Bmatrix} \psi_{x,x} \\ \psi_{x,y} \\ \psi_{y,x} \\ \psi_{y,y} \end{Bmatrix} dx\, dy \tag{10}$$

In order to complete the formulation the kinetic energy expression, must be written in terms of the displacement field. Again, assuming symmetry with respect to the z axis, the kinetic energy can be put in terms of the midsurface displacements and mass related quantities,

$$T = \frac{1}{2}\int_A \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{\psi}_x \\ \dot{\psi}_y \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{\psi}_x \\ \dot{\psi}_y \end{Bmatrix} dx\, dy \tag{11}$$

The mass related terms in Eqn (11) are defined by

$$(m, I) = \int_{z_0}^{z_k} \left(l, z^2\right) \rho\, dz \tag{12}$$

and they represent respectively the plate mass and moment of inertia per unit area.

2.3 - Finite element formulation

A finite element formulation was developed to solve both the inplane stress problem and the free vibration problem described in the previous sections. A 16-node isoparametric bi-cubic element using standard Lagrangian interpolation functions was implemented in a FORTRAN code. This simple element does not suffer from shear locking, being therefore applicable to thick as well as to thin laminates (Heppler & Hansen, 1986). No special techniques such as reduced integration need to be applied for the integration of the stiffness matrix; a regular 16-point Gaussian quadrature scheme is used. A shear correction factor of 5/6 is adopted for all laminates.

The element has five degrees of freedom per node (u, v, w, $\psi_{,x}$ and $\psi_{,y}$) totalizing 80 displacement degrees of freedom. The displacements at an arbitrary point within the domain of the p-th element are written in terms of element nodal displacements as $\{\delta\}^T=[N]\{\delta\}_p$, where $[N]$ is a 5×80 matrix of interpolation functions. Vector $\{\delta\}_p$ contains the 80 element nodal displacement degrees of freedom arranged by nodes.

The piezoelectric induced inplane stress problem may be solved from the stationary value of Eqn (5), resulting:

$$[K]\{\delta\} = \{F\} \tag{13}$$

in terms of the system nodal displacements $\{\delta\}$ The element stiffness matrix, [K], and electric force vector, {F},may be found from Eqn. (5) using standard finite element texhniques. The inplane stress resultants can then be calculated from Eqn. (7).

Applying Hamilton's principle yields the equations of motion for the plate under free vibration. Again, global matrices consistent with the boundary conditions of the problem can then be generated using standard finite element techniques. The resulting equation can be cast in the form of an eigenvalue problem in terms of the vector of system nodal displacements $\{\delta\}$ as

$$\left[[K]+[K_G]-\omega^2[M]\right]\{\delta\}=\{0\} \tag{14}$$

where [M], [K], and [K_G] are the global mass, stiffness and geometric stiffness matrix for the plate. The geometric stiffness matrix may be derived from Eqn. (10). In this equation, it can be recognized that the first term corresponds to the classical Kirchhoff plate theory while the second is related to Mindlin plate hypothesis.

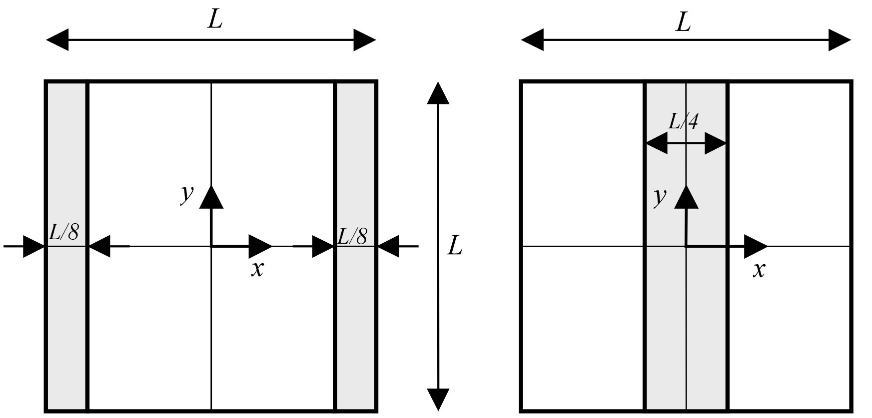
3. NUMERICAL SIMULATIONS

In order to illustrate the effect of the piezoelectrically induced stress stiffening on the mechanical behavior of a composite plate, a graphite epoxy plate with PZT actuators is considered. The plate is unconstrained and square with length equal to *360mm*. The basic laminate for the analysis has four graphite/epoxy unidirectional tape layers, with the stacking sequence $[0/90]_s$ where the 0^o corresponds to the x global axis. The material properties used in the analysis are listed in Table 1.

Table 1 - Physical properties of the materials used

Property	PZT	Graphite/Epoxy
Young´s moduli, E_{11} (GPa)	63.0	154.0
Young´s moduli, E_{22} (GPa)	63.0	11.13
Poisson´s ratio, ν_{12}	0.3	0.304
Shear moduli, G_{12} (GPa)	24.2	6.98
Shear moduli, G_{13} (GPa)	24.2	6.98
Shear moduli, G_{23} (GPa)	24.2	3.36
Laminate thickness (mm)	0.254	0.150
Density, ρ (Kg/m^3)	7600	1560
PZT constants, e_{31} (N/V mm)	0.0229	—
Breakdown voltage (V/mm)	1000	—

Two different geometric arrangements of piezoelectric patches are considered in order to assess the effects of the inplane induced stress resultants in the natural vibration frequencies of a square plate. Besides inducing inplane stress, piezoelectric elements affect the plate distribution of mass and stiffness. Symmetry with respect to the z axis exists, since identical piezoelectric patches are symmetrically bonded to the top and bottom surfaces. Figure 1 depicts the proposed configurations, which are similar to those used to investigate the stress stiffening effects due to thermal residual stresses (Almeida & Hansen, 1997).



(a) longitudinal actuators at the edges (b) longitudinal actuators at the center

Figure 1 - Piezolectric actuator configurations

The first configuration, depicted in Fig. 1a, has two longitudinal actuators along the edges of the plate. In a second configuration, Fig. 1b, the actuators are also longitudinal and have the same dimensions of the first case but they are placed at the center of the plate.

Performing a punctual sweeping in the value of the voltage, distinct eigenvalue problems were solved for each of the four mentioned plate configurations. Furthermore, the voltage which is the same for all piezoelectric material patches obeys the appropriate material breakdown voltage limits (*254V*). The results were used to plot charts of voltage versus the natural frequency of vibration shown in the Figs. 2 and 3. In both cases presented the inplane stress resultants are assumed to be proportional to the applied voltages. Therefore, previously to the solution of the eigenvalue problems for a given configuration of PZT patches over a given plate, only one linear static analysis was performed in order to calculate the induced inplane stress distribution resulting from piezoelectric actuation.

Due to plate symmetry vibration modes are either symmetric or antisymmetric respectively to the x and y plate axis, allowing the discretization of only one-fourth of the plate with a uniform (4×4) rectangular mesh of 16 elements. Appropriate boundary conditions were then applied to the nodes of the discretized plate, such that four different finite element models were created. Model SS accounts for vibration modes that are symmetric with respect to the x and y coordinate axes, while model AA represent modes which are antisymmetric with respect to both axis. Model SA represents modal symmetry with respect to the x axis and antisymmetry with respect to the y axis, while SA represents symmetry in x and antisymmetry in y.

4. DISCUSSION

The numerical results demonstrate that the natural vibration frequencies of a plate equipped with two different configurations of piezoelectric patches are significantly influenced by induced stress stiffening effects. In fact, plate behavior is rather complex, since in general induced tension is not associated with stiffening nor induced compression is associated with the softening of the four types of plates. Furthermore, it is apparent that the configuration of the piezoelectric material patches is decisive in the vibration behavior characteristics of the plates studied.

Figure 2 shows the plots of natural frequency versus voltage for the first five modes for the plate with two longitudinal actuators placed at the edges. The two fundamental natural frequencies are very close in the neutral state ($V_o = 0\ V$), but tend to separate from each other with the decrease in voltage or increase beyond the value for which both frequencies become the same. While the fundamental frequency (first SS mode) increases with tension stress the second mode (first AA mode) has its frequency decreasing, such that the increase in tension stress leads to the switch between the two modes. Notice that for higher positive or negative values of voltage the plate aproaches its critical load as the fundamental frequencies tend to zero. For the third (SS) and fourth (AS) modes the increase in induced compressive stress tend to stiffen the plate, as is the case for the second mode (first AA mode). For the fifth mode the natural frequency reaches a maximum for a voltage around -90V.

Figure 3 depicts the frequencies of a plate with two longitudinal actuators placed at the center. Here the fundamental mode is less sensitive to changes in voltage and the frequency reaches a maximum value for a voltage close to the neutral state. Similarly to the plate in Fig. 2, there is a switch in the fundamental modes, but now due to the increase of compressive induced stress; the third and higher modes tend however to stiffen with tension. Comparing Figs. 2 and 3 it is concluded that the stress stiffening effect of the piezoelectric inplane stresses strongly depends, among several other factors, on the geometrical arrangement of the actuators on the plate.

5. CONCLUSIONS

A study of the stress stiffening effects on the free vibration behavior of composite plates with piezoelectric actuators was conducted using a consistent finite element formulation based on Reissner-Mindlin plate theory. The numerical results presented for the case of unconstrained plates show that the piezoelectric induced inplane stress effect may be significant for the free vibration behavior. The relative importance of the stress stiffening effect depends on the magnitude of the in-phase actuation and geometric arrangement of the piezoelectric actuators; boundary conditions; geometry of the problem; and material properties.

Only two simple geometric configurations of piezoelectric patches were tested in the examples; obviously several other configurations would be possible, but our purpose was to illustrate the stress stiffening effects using a set of simple examples. The same thinking guided us towards using the same actuation voltage for the piezoelectric elements, although it would be a fairly simple matter to exploit the use of several different voltages for the piezoelectric patches. In more realistic applications the configuration of the piezoelectric patches as well as the voltages may be tailored to achieve a certain predefined performance criterion. It seems natural that structural optimization would be a powerful tool to fit very adequately the design of smart systems considering inplane induced stress stiffening, since plate behavior seems difficult to be captured through intuition only.

The ability of controlling the plate stiffness has potentially interesting applications, one of them being the tuning of the frequencies of vibration of smart composite structures.

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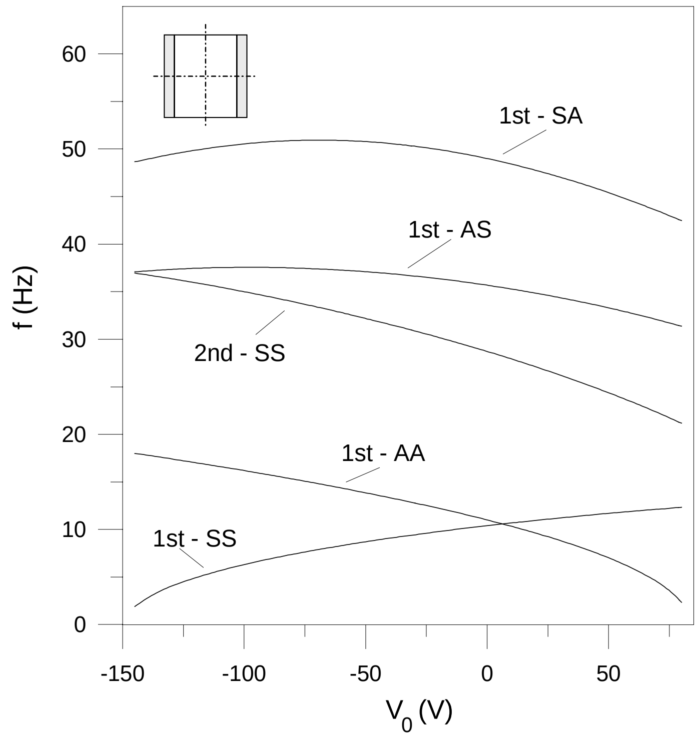

Figure 2 - Natural frequency as a function of voltage for longitudinal actuators at the edges

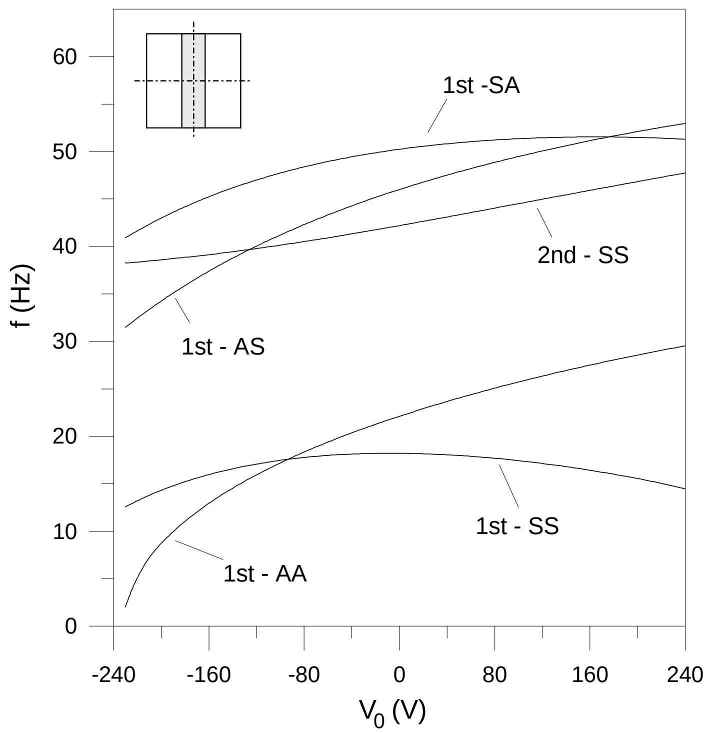

Figure 3 - Natural frequency as a function of voltage for actuators at the center

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