COBEM 99
XV Congresso Brasileiro de Engenharia Mecânica
15th Brazilian Congress of Mechanical Engineering
22 - 26 de Novembro de 1999 / November 22 - 26, 1999 Águas de Lindóia, São Paulo,

STRUCTURAL DYNAMICS OF PRE-LOADED DOUBLY CURVED SHALLOW SHELLS

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ABSTRACT. *A variational formulation of pre-loaded doubly curved shells is presented. The analysis uses a two-field variable principle with w, the vertical tranverse displacement and F, Airy stress function considered as the field variables. Finite elements satis fying* C^1 *continuity are used for the solution of the problem. Several applications are presented and the results are discussed and are compared with previous analytical, numerical and experimental works.*

1. INTRODUCTION

With the advent of high speed digital computation, the finite element method turned to be the most adequate and accurate means of structural analysis of complex structures. Since the introduction of the element conistent mass concept by Archer in 1963, many authors extented the application of the method to structural dynamics response problems. The structural dynamic analysis of plates and shells has usually been performed using the Hamilton's principle with the displacements *u,v* and *w* taken as the field variables of the problem. Alternatively, the problem can be formulated using a two field variable modified functional with the transverse displacement *w*, and Airy stress function *F*, as the field variables of the problem. Bismark-Nasr (1993-c), presented a formulation using the transverse displacement *w*, and Airy stress function *F*, as the field variables for the problem of free vibration analysis of isotropic cylindrically curved shallow shells. The Euler-Lagrange equations governing the problem and the boundary conditions were obtained. It was shown that the boundary conditions on *F* are as simple and direct to apply as on *w*. Bismarck-Nasr (1993-b) and (1994), extented then the application to the buckling analysis of isotropic cylindrically curved plates, and in (1993-d), to the supersonic flutter of cylindrically curved isotropic panels. Further, in (1995-a), Bismarck-Nasr, treats the problem of stability of cantilever cylindrically curved isotropic panels subjected to nonconservative tangential follower forces distributed over the area of the panel and a distributed follower at the free end of the panel using a two field variational formulation. Free vibration analysis of laminated doubly curved shallow shells using a two field variational formulation, were presented, by Bismarck-Nasr and Silva (1995-b). Aeroelasticity and buckling analysis of laminated fibre-reinforced doubly curved shallow shells were presented by Bismarck-Nasr in 1998a and 1998b using a two field variational formulation. The purpose of the present work is to present a two-field variable variational formulation, with *w* and *F* taken as the field variables, for the structural dynamic response problems of pre-loaded doubly curved isotropic shallow shells. The solution of the problem is made using a C^1 continuity finite element method. Numerical results are given and the results obtained are discussed and are compared with previous solutions.

2. PROBLEM FORMULATION

Following the formulation, given by Bismarck-Nasr (1991), (1993-a,b) and (1994), and including the effect of the curvature of the shell in the transverse direction, the variational equation of an isotropic doubly curved shallow shells, considering the effect of the work done by an initial pre-load inplane prestress load, N_x^0 ,N_y^0 ,and N_{xy}^0 , can be written as,

$$\delta(\Pi^*) = \delta\int_{t_1}^{t_2} \Big[\Big[\int_A \rho h w_{,tt}\, dA - \int_A w \Big\{ \frac{F_{,xx}}{R_x} + \frac{F_{,yy}}{R_y} \Big\} dA$$

$$- \frac{D}{2}\int_A \big[w^2_{,xx} + w^2_{,yy} + 2\nu w_{,xx} w_{,yy} - 2(1-\nu) w^2_{,xy} \big] dA$$

$$+ \frac{1}{2Eh}\int_A \big[F^2_{,xx} + F^2_{,yy} - 2\nu F_{,xx} F_{,yy} - 2(1+\nu) F^2_{,xy} \big] dA$$

$$+ \frac{1}{2}\int_A \big[N_{xx}^0 w_{,x}^2 + N_{yy}^0 w_{,y}^2 + 2N_{xy}^0 w_{,x} w_{,y} \big] dA \Big] dt = 0$$

(1)

where the functions subjected to variation are the transverse displacement *w* and the Airy stress function *F*, $D = Eh^3/12(1-\nu^2)$, *E* is Young's modulus, *h* is the shell thickness and ν is Poisson's ratio. The Airy stress function is defined as,

$$N_{xx} = \frac{\partial^2 F}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

(2)

Performing the variational operation in Eq. (1), the Euler-Lagrange equations governing the problem are obtained and read,

$$\rho h w_{,tt} + D\nabla^4 w + \frac{1}{R_x} F_{,xx} + \frac{1}{R_y} F_{,yy} - N_{xx} w_{,xx} - N_{yy} w_{,yy} - 2N_{xy} w_{,xy} = 0$$

$$\nabla^4 F - \frac{Eh}{R_x} w_{,xx} - \frac{Eh}{R_y} w_{,yy} = 0 \tag{3}$$

and the boundary condition for an edge ν = constant are given by: 1. Clamped edges $w = w_{,\nu} = 0$, and at a corner $F_{,\mu\nu} = 0$, 2. Free edges $F = F_{,\nu} = 0$, and at a corner $M_{,\mu\nu} = 0$, (i.e., $w_{,\mu\nu} = 0$), 3. Simply supported edges $w = 0$, and at a corner $F_{,\mu\nu} = 0$ and 4. Freely supported edges $w = F = 0$. A finite element method for the solution of the problem at hand can be performed using rectangular elements that preserve C^1 continuity, by writing for the functions w and F interpolation functions in term of the nodal parameter as,

$$z(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} [\, H_{0i}(x) H_{0j}(y)\, z_{ij} + H_{1i}(x) H_{0j}(y)\, z_{,x_{ij}} + H_{0i}(x) H_{1j}(y)\, z_{,y_{ij}} + H_{1i}(x) H_{1j}(y)\, z_{,xy_{ij}} \,] \tag{4}$$

where z stands for w or F, and H_{mn} are first order Hermitian polynomials. Using the standard finite element technique we obtain for each element a set of two equations cast in the form,

$$[M]\{w''\} + [k_{ww}]\{w\} + [k_{wF}]\{F\} + \left[N_{xx}[k_{G_{Nxx}}] + N_{yy}[k_{G_{Nyy}}] + N_{xy}[k_{G_{Nxy}}] \right] \{w\} = 0$$

$$[k_{Fw}]\{w\} + [k_{FF}]\{F\} = 0 \tag{5}$$

The element stiffness matrix $[k_{ww}]$, compatibility matrix $[k_{FF}]$ and the geometric matrices are the same as given by Bismarck-Nasr, (1993-b). The elements of the coupling matrix $[k_{wF}]$ read,

$$k_{wF_{ij}} = \frac{1}{R_y} S1_a(n_j, n_i)\, R2_b(m_i, m_j) + \frac{1}{R_x} S1_b(n_j, n_i)\, R2_a(m_i, m_j) \tag{6}$$

where the (4x4) matrices $S1_a$ and $R2_b$ are given in Bismarck-Nasr (1991). The element mass matrix is given in Bismarck-Nasr (1991). Using now the standard finite element assembly technique and applying the boundary conditions, we obtain for the whole structure the following two matrix equations,

$$[M]\{w''\} + [K_{ww}]\{w\} + [K_{wF}]\{F\} + \left[N_{xx}[K_{G_{Nxx}}] + N_{yy}[K_{G_{Nyy}}] + N_{xy}[K_{G_{Nxy}}] \right]\{w\} = 0$$

$$[K_{Fw}]\{w\} + [K_{FF}]\{F\} = 0$$

(7)

We observe that the degree of freedom $\{F\}$can be eliminated using the compatibility equation of the system of equations, i.e., the second equation of the system (7), to obtain,

$$[M]\{w''\} + [K_{eq}]\{w\} + \left[N_{xx}[K_{G_{Nxx}}] + N_{yy}[K_{G_{Nyy}}] + N_{xy}[K_{G_{Nxy}}] \right]\{w\} = 0$$

(8)

where,

$$[K_{eq}] = [K_{ww}] - [K_{wF}][K_{FF}]^{-1}[K_{Fw}]$$

(9)

Examination of eq. (9) reveals that the computational effort required for the solution of the structural dynamic response problem when the present formulation is used is equivalent to that of a flat plate. Further, the inplane boundary conditions are applied on F , $F_{,x}$,$F_{,y}$ and $F_{,xy}$ and are all nodal degrees of freedom of the finite element model.

3. NUMERICAL RESULTS AND DISCUSSIONS

The present formulation permits the studies of structural dynamic response problems of pre-loaded plates and shallow shells. As special cases are included the free vibrration analysis of plates and shallow shells and the buckling analysis analysis of plates and shallow shells. In the following some of the results obtained using the present formulation are reported and are compared with previous investigation whenever possible.

The first example reported is a buckling analysis of square clamped plates on all edges. table 1 reports the results obtained using the present formulation and these are compared with previous analytical solutios and other finite element formulations.

The second example treats the determination of the buckling coefficients of cylindrically curved shallow shells for freely supported and clamped boundary conditions on all edges and these results are compared the analytical solutions given in Bruhn (1973). From the results presented in both Tables 1 and 2, it can be observed that good accuracy was obtained using the present formulation with only a mesh size of 4 by 4 elements.

Table 1. Buckling coefficient, $N_{cr} = N_{xx}a^2/\pi^2D$ for all edges clamped plates.

Reference	ANAL. Timoshenko 1961	FEMT Allman 1971	FEMT Clough 1968	FEMR Kapur 1966	FEMR Dawe 1969	FEMR Carson 1969	FEMR Present
Load							
Axial	10.07	10.99	-	9.284	10.147	-	10.192
bi-axial	5.30	5.602	5.625	4.975	-	5.3271	5.326
shear	14.71	17.382	-	-	-	15.043	15.122

FEMT = triangular finite element , FEMR = rectangular finite element

Table 2. Shear buckling coefficient, $N_{cr} = N_{xy}a^2/\pi^2D$ for all edges clamped cylindrically curves shallow shells. h= 0.05 cm and R= 400 cm.

Reference	ANAL. Bruhn 1973	FEM Present	ANAL. Bruhn 1973	FEM Present
a/b	Freely-supported		Clamped	
1.0	6.3084	6.6149	8.4126	8.7890
1.5	4.7321	5.0526	8.0620	8.4789
2.0	4.2063	4.6902	7.3710	7.9688
3.0	3.8558	4.3658	6.3094	6.6514

The next series of calculation presented are for free vibration analysis of freely supported on all edges spherically curved shallow shells. Table 3 gives the results obtained using the present formulation and these are compared with the Reissner analytical solution reported on 1955. From the results obtained, it can be observed that good accuracy was obtained using the present formulation with only a mesh size of 4 by 4 elements.

The last series of results presented are for free vibration analysis of cylindrically curved shallow shells for freely supported conditions on all edges in the presence of axial pre-load in the axial direction. The analysis was performed for different pre-load conditions and the related natural frequencies were obtained. It is to be observed the the critical loading condition is obtain as a subproduct of this analysis and is reached when a zero frequency is observed. The results obtained are reported in Table 4.

Table 3. Natural frequency parameter, $\Omega = \rho h a^4 \omega^2/\pi^2 D$ for all edges freely supported, spherically curved shallow shells, a/b=1, $1/R$ = 0.0005 cm^{-1}.

Mode	*m*	*n*	Reissner 1955 Analytical	Present FEM
1	1	1	2.76444	2.80532
2	1	2	5.35183	5.47992
3	2	1	5.35183	5.47992
4	2	2	8.22448	8.22888
5	1	3	10.18048	10.33221
6	3	1	10.18048	10.33221
7	2	3	13.13933	13.50552
8	3	2	13.13933	13.50552
9	1	4	17.10679	17.44990
10	4	1	17.10679	17.44990

m,n are the number of halfsine waves in the *x,y* direction respectively.

Table 4. Fundamental natural frequency parameter, $\Omega = \rho h a^4 \omega^2/\pi^2 D$ for all edges freely supported, cylindrically curved shallow shells, in the presence of an axial pre-load, $N^* = N_{xx}a^2/\pi^2 D$, a/b=1, $1/R$ = 0.02 cm^{-1}.

N^*	Ω	
0	12.8	
10	11.9	
20	10.8	
30	9.4	
40	8.1	
50	6.3	
60	4.2	
66.4	0.0	buckling load

CONCLUSIONS

Structural dynamic response problems of pre-loaded doubly curved isotropic shallow shells has been presented. The analysis is based on a modified two field variable variational principle. Numerical results are given and the results obtained are discussed and are compared with previous solutions, whenever availables. The analysis presented permits the free vibration analysis, combined buckling loads and structural dynamic responce in the presence of pre-loads.

ACKNOWLEDGMENTS

Grant 300954/91-3 of CNPq (Brazil) conceded to the first author and CAPES support (Brazil) to the second author during the preparation of this work are gratefully acknowledged.

REFERENCES

ALLMAN, D.J., (1971) Finite element analysis of plate buckling using a mixed variational principle, Proc. 3rd. Conf. Matrix Meths. Mech., Wright Patterson AFB, Ohio, USA.

ARCHER, J.S., (1963) Consistant mass matrix for distributed systems, Proc. Amerc. Soc. Civil Engng., Vol. 89, No. ST4, pp. 161-178.

BISMARCK-NASR, M.N., (1991) On the sixteen degree of freedom rectangular plate element, *International Journal of Computers and Structures,* Vol.40, No. 4, pp. 1059-1060.

BISMARCK-NASR,M.N., (1993-a) *Finite Elements in Applied Mechanics,* Abaeté Gráfica, São Paulo.

BISMARCK-NASR, M.N., (1993-b) Analysis of Cylindrically Curved Panels Based on a Two Field Variable Variational Principle, *Applied Mechanics Reviews,* Vol. 46, No. 11, Part 2, pp. 571-578.

BISMARCK-NASR, M.N., (1993-c) On Vibration of Thin Cylindrically Curved Panels, *Proc. Third Pan American Congress of Applied Mechanics, PACAM III,* São Paulo, pp. 696-699.

BISMARCK-NASR, M.N.,(1993-d) Supersonic Panel Flutter Analysis of Shallow Shells, *AIAA Journal,* Vol. 31, No. 7, pp. 1349-1351.

BISMARCK-NASR, M.N., (1994) Buckling Analysis of Cylindrically Curved Panels Based on a Two Field Variable Variational Principle, *International Journal Computers and Structures,* Vol. 51, No. 4, pp. 453-457.

BISMARCK-NASR, M.N., (1995-a) Dynamic Stability of Shallow Shells Subjected to Follower Forces, *AIAA Journal,* Vol. 33, No. 2, pp. 355-360.

BISMARCK-NASR, M.N., and SILVA D.A.H., (1995-b) Vibration of Laminated Doubly Curved Shallow Shells, *Proc. Fourth Pan American Congress of Applied Mechanics, PACAM IV,* Buenes Aires, pp. 337-342.

BISMARCK-NASR, M.N., (1998-a) Aeroelasticity of laminated fibre-reinforced doubly curved shallow shells, *AIAA Journal,* Vol. 36, No. 4, pp. 661-663.

BISMARCK-NASR, M.N., (1998-b) Buckling analysis of laminated fibre-reinforced doubly curved shallow shells, *Journal of aircraft*, Vol. 35, No. 3, pp. 502-505.

BRUHN, E.F., (1973) *Analysis and design of flight vehicle structures*, Tri-State Offset Company, Cincenatti, USA.

CARSON, W.G., & NEWTON, R.E., (1969) Plate buckling analysis using a fully compatible finite element, *AIAA J.*, Vol. 7, No. 3, pp. 527-529.

CLOUGH, R.W., and FELIPA, C.A., (1968) A refined quadrilateral element for analysis of plates bending, Proc. 2nd. Conf. Matrix Meths. Mech., Wright Patterson AFB, Ohio, USA.

DAWE, D.J., (1966) Application of the discrete element method to the buckling analysis of rectangular plates under arbitrary membrane loading, *Aeron. Quart.*, Vol. 20, pp. 114-128.

KAPUR, K.K., & HARTZ, J., (1966) Stability of plates using the finite element method, *J. Engng. Mech. Div.*,Vol. 92, No. EM2, pp. 177-195.

REISSNER, E., (1955) On transverse vibration of thin shallow elastic shells, *Quart. Applied Maths.*, Vol.13, No. 2., pp. 169-177.

TIMOSHENKO, S.P., and GERE, J.M., (1961) Theory of elastic stability, McGraw-Hill, 2nd Ed., New York, USA.