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# The Use of Convex Sets in Damage Detection

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***Abstract.*** Damage detection in a cantilever Euler-Bernoulli beam is investigated. The mathematical model for the system is obtained using the Finite Method and static condensation. In this paper, damage is considered a reduction in the stiffness of any particular element of the model. The detection scheme is based on establishing failure sets which predict the natural frequency changes in the beam due to damage at a given element location. These failure sets are compared with the actual beam's response using an Euclidean norm. The model is restricted to a conservative system that is described by a positive definite matrix capable of being transformed into a diagonal Jordan canonical form. The restriction of the mathematical model enables the formulation of a standard eigenvalue problem. This representation has a geometric interpretation of a n-dimensional ellipsoid that is a convex set. An example of a two-degree of freedom model is established to examine the effect damage has on the geometric representation of the eigenvalue problem. The geometric representation of the eigenvalue problem of the damaged state is the convex hull of the beam's previous existing condition. This leads to the mathematical validation of the above mentioned damage detection scheme. Experimental verification is performed for several cases of the simplified model. The results show that the method is easy to use and does not require a rigorous amount of instrumentation for obtaining the experimental data required by the damage detection scheme.

***Keywords***: Damage detection, Convex sets, Euler-Bernoulli beam

## 1. Introduction

Damage can cause the properties of structures and their elements to undergo changes. Serval researchers, have noted that damage which decreases the structural rigidity, tends to decrease the natural frequencies of the structure,( *Adams R. D. et al. 1978, Bastadzhan A. Z. et al. 1990, Dimarogonas A. S. et al. 1983, Rizos P.R. et al. 1990,* and *Yuen M. F. 1985).* One of the first studies in vibration response measurement was performed by *Adams et al. 1970.* Here the study consisted of measuring the longitudinal vibration response of various structures and determining the change in the natural frequencies. *Rizos et al 1990*, studied the vibration of a cantilever beam with a crack. In their analysis the vibration response at two arbitrary locations was measured along with the frequency of vibration. These results were used to determine the crack depth and location Many other methods have been performed in damage detection and are too numerous to mention in this small introduction to the subject. This paper will examine the effect of damage on the geometric interpretation of the eigenvalue problem in an arbitrary two degree of freedom structure and verify the method with a simple case.

## 2. Model: General Development and Form

In vibration modeling, equations are usually obtained from discrete methods such as the finite element analysis or lumped parameter models to name a few. These models often consist of a set of coupled ordinary differential equation which can be express in a general matrix form given as:

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{f(t)\} \tag{1}$$

The generalized coordinates used to describe the motion is represented by the vector $\{y\}$. The matrices $[M], [C]$, and $[K]$ are used to describe the mass, damping, and stiffness of the structure respectively. The generalized forces acting on the structure, are described by the vector $\{f(t)\}$. These equations of motion can also be derived from the virtual work associated with nonconservative forces and two scalar functions, the kinetic energy and potential energy. This discussion will be limited to natural conservative systems with proportional damping. In addition, Eq. (1), represents the linearized governing equations about an equilibrium point.

## 3. The Eigenvalue Problem

The eigenvalues corresponding to the system of equations described by Eq. (1), may be characterized by the properties of the mass, damping and stiffness matrices. A common method used to solve these equations is the method of modal analysis which requires the solution of the eigenvalue problem for the system. The eigenvalue problem is developed from the homogeneous representation of Eq. (1) in which damping has been neglected. This us usually expressed in the form:

$$[M\}\{\ddot{y}\} + [K]\{y\} = 0. \tag{2}$$

The eigenvalue problem in this discussion will be given in a standard form representation as

$$[A]\{v\} = \lambda\{v\}. \tag{3}$$

The property of symmetry, given in the matrices $[K]$ and $[M]$, is desired to be retained in the matrix $[A]$. In order to accomplish this, the particular linear transformation

$$\{y\} = [M]^{\frac{1}{2}}\{v\}$$

will be used. Substitution into Eq. (2), assuming synchronous motion with an exponential solution and performing the necessary algebra yields Eq. (3). Matrix $[A]$ is defined by

$$[A] = [M]^{-\frac{1}{2}}[K][M]^{-\frac{1}{2}}$$

and $\lambda$ is used to represent the eigenvalue of the system. Matrix $[A]$ is a symmetric matrix whose form is defined from the pre and post multiplication of a symmetric positive definite matrix, $[M^{-\frac{1}{2}}]$. The properties of matrix $[A]$ being positive definite or positive semi definite is determined by matrix $[K]$. The matrix $[K]$, which is usually considered positive semi definite, is determined by the potential energy function, $V$. Only a potential energy function derived from elastic forces will be considered in this discussion.

### 3.1 Geometric Interpretation of the Eigenvalue Problem and Convexity

In order to obtain an understanding of the geometric interpretation of the eigenvalue problem, a brief review of quadric surfaces will be presented. A quadric surface is defined as the subset $F$ in $E^3$ such that $F$ is the set of zeros of the quadratic equation $\mathrm{G}(u, v, w) = 0$. The variables $u, v,$ and $w$ represent a set of arbitrary rectangular coordinates located in $E^3$. The general equation for the quadric surface may be written as

$$au^2 + bv^2 + cw^2 + d + 2a'vw + 2b'uw + 2c'uv + 2a''u$$

$$+ 2b''v + 2c''w = 0 \tag{4}$$

in which the coefficients $a, b,\ c, d, a', b', c', a'', b''$, and $c''$ are real numbers. When the quadric surface is central, meaning it has a center, and if Eq. (4) contains no linear terms, the surface can be represented by the following form

$$\{u \quad v \quad w\} \begin{bmatrix} a & c' & b' \\ c' & b & a' \\ b' & a' & c \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = -d = d'. \tag{5}$$

If the matrix is positive definite and a suitable rectangular coordinate system, $(\mathcal{U},\mathcal{V},\mathcal{W})$, is used to describe the surface, Eq. (5) can be expressed as

$$\frac{\mathcal{U}^2}{\mathcal{A}^2} + \frac{\mathcal{V}^2}{\mathcal{B}^2} + \frac{\mathcal{W}^2}{\mathcal{C}^2} = 1 \tag{6}$$

in which $\mathcal{A}, \mathcal{B},$ and $\mathcal{C}$ are constants. Equation (6) describes an ellipsoid with a rectangular coordinate system located at its center. For a 2-dimensional Euclidian space, Eq. (6) reduces to an ellipse.

Quadric surfaces may be extended to a $n$-dimensional Euclidean space, $E^n$, without loss of generality. The equations which were used to describe the ellipsoidal surface now contain $n$ terms. A quadratic surface in $\mathrm{E}^n$ possessing the requirements for an ellipsoid description (no linear terms and a real symmetric positive definite matrix representation) is referred to as an elliptic quadratic hypersurface. The significance of this is that an $n \times n$ matrix equation, $n \geq 4$, can be used to represent this particular hypersurface.

The eigenvalue problem of a natural conservative system is a central quadric hypersurface which has the property of convexity. A region is said to be convex if a line segment joining any two points in the region is located entirely within the region. Consider the two distinct regions which are represented by the closed sets shown in Fig. 1. The geometric property of convexity is easily viewed in Figure 1(a). As long as the points $p$ and $q$ are with in the region, the line segment $\overline{pq}$ will be in the region. However, this is not the case for the non convex region shown in Figure 1(b). Even though the points $p'q'$ are contained in the region, part of the line segment $\overline{p'q'}$ is located outside the set.

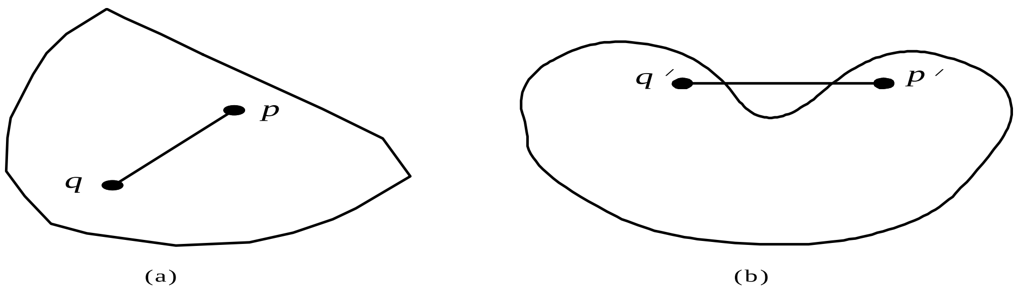


Figure 1: Two regions containing line segments in a 2-dimensional Euclidean space.

In a 2-dimensional Euclidean space, as shown in Fig. 1, the convexity of a set can easily be determined. When sets become more complex and are harder to visualize, an algebraic formulation is required and may be given as

$$Let\,S\,be\,a\,set\,of\,points\,in\,an\,n-dimensional\,Euclidan\,space\,E^n.$$
$$For\,any\,two\,points\,p\,and\,q\,such\,that\,p\in S\,and\,q\in S\,there\,exists\,a$$
$$series\,of\,points\,in\,S\,defined\,such\,that$$

$$\kappa p+(1-\kappa)q\in S$$

$$for\,all\,values\,of\,\kappa\,within\,the\,range\,of\,0\leq\kappa\leq 1.\;S\,is\,defined\,as\,a$$
$$convex\,set\,if\,all\,averages\,of\,S,\;as\,defined\,by\,the\,above\,equation,$$
$$belong\,to\,S.$$

This algebraic definition can be used for $n$-dimensional regions and is equivalent to the geometric presentation which has been given. Convex sets are usually described by the convex hull of the set which is by definition the intersection of all convex sets containing the region described by the set. Another way to interpret this definition is to say that a convex hull is the smallest convex set describing the region. By using the definition of a convex set and convex hull, the convexity of the geometric interpretation of the eigenvalue problem can be shown to be *Just* (1997)

$$\kappa\,\overrightarrow{p}^{\mathrm{T}}A\overrightarrow{p}+(1-\kappa)\,\overrightarrow{q}^{\mathrm{T}}A\overrightarrow{q}\leq 1.$$

The geometric interpretation of the eigenvalue problem, an elliptic quadric hypersurface, is always a convex set in any $n$-dimensional Euclidean space.

**4. An Arbitrary Two-Degree of Freedom System**

An arbitrary lumped mass two degree of freedom system, shown in Fig. 2, will be examined in terms of its convexity.

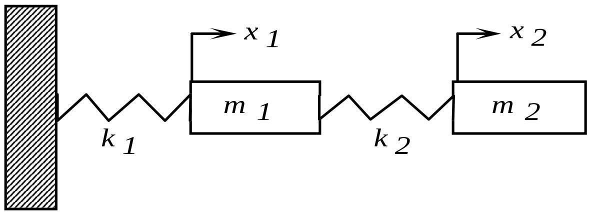


Figure 2: A Lumped Mass Two Degree of Freedom System.

The displacements $x_1$ and $x_2$ describe the motions of the lumped masses $m_1$ and $m_2$ respectively. The stiffness of this system is represented by the linear springs $k_1$ and $k_2$. The potential energy function , $V$, is given by

$$V = \frac{1}{2}k_1(x_1)^2 + \frac{1}{2}k_2(x_2 - x_1)^2$$

and the equilibrium point is given by $x_1 = 0$ and $x_2 = 0$. The linearized equations of motion about the equilibrium point yields

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0. \tag{7}$$

The solution of this equation requires the construction of the eigenvalue problem which can be expressed in a standard form as

$$\begin{bmatrix} \frac{(k_1+k_2)}{m_1} & \frac{-k_2}{\sqrt{m_1 m_2}} \\ \frac{-k_2}{\sqrt{m_1 m_2}} & \frac{k_2}{m_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}. \tag{8}$$

As discussed before, the geometric interpretation of this problem is equivalent to finding the principal axes of an ellipse. The equation for this curve can be given by

$$\frac{(k_1 + k_2)}{m_1} u_1^2 - 2\frac{k_2}{\sqrt{m_1 m_2}} u_1 u_2 + \frac{k_2}{m_2} u_2^2 = 1. \tag{9}$$

Damage, which is considered a reduction of the stiffness in the first and/or second element, can be represented by the parameters $\alpha$ and $\beta$ respectively. Equations (8) and (9) can be transformed repectively to

$$\begin{bmatrix} \left(\alpha k_{11} + \beta \frac{m_2}{m_1} k_{22}\right) & -\beta k_{12} \\ -\beta k_{12} & \beta k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{10}$$

and

$$\left(\alpha k_{11} + \beta \frac{m_2}{m_1} k_{22}\right) u_1^2 - 2\beta k_{12} u_1 u_2 + \beta k_{22} u_2^2 = 1 \tag{11}$$

in which the terms $k_{11}$, $k_{12}$, and $k_{22}$ are defined by $k_{11} = \frac{k_1}{m_1}$; $k_{12} = \frac{k_2}{\sqrt{m_1 m_2}}$, and $k_{22} = \frac{k_2}{m_2}$.

Equation (10) defines an ellipse whose coefficients contain the parameters used to describe damage. The effect of the various possible damage scenarios, can be simulated by substituting different values for $\alpha$ and $\beta$ into Eq. (10). Each scenario will correspond to a particular ellipse. As the various damage scenarios are studied, the domain described by the boundary of the ellipse changes. The behavior of these ellipses can be studied by using two forms of Eq. (10). One form will correspond to the case in which no damage has occurred in the system ($\alpha = 1$, $\beta = 1$). The other will represent the system with any arbitrary state of damage. These conditions are represented by

$$\left(k_{11} + \frac{m_2}{m_1}k_{22}\right)u_1^2 - 2k_{12}u_1u_2 + k_{22}u_2^2 = 1 \tag{12}$$

and

$$\left(\alpha k_{11} + \beta\frac{m_2}{m_1}k_{22}\right)u_1^2 - 2\beta k_{12}u_1u_2 + \beta k_{22}u_2^2 = 1. \tag{13}$$

A damage detection scheme can be established if the effect of damage on the ellipse produces a convex hull of the apriori condition. This will occur when the boundaries defining the convex sets are tangent or never intersect each other. The apriori condition and the condition corresponding to the damaged system are defined by Eqs. (12) and (13) respectively.

For an intersection to exist, points from both curves must simultaneously satisfy Eqs. (12) and (13). From Eq. (13), the variable $u_2$ can be expressed as a function of the variable $u_1$.

$$u_2 = \frac{1}{2\beta k_{22}}\left(2\beta k_{12}u_1 \pm 2\sqrt{-\beta\left(\frac{-\beta k_{12}^2u_1^2m_1 + k_{22}u_1^2\alpha k_{11}m_1 + u_1^2\beta m_2k_{22}^2 - k_{22}m_1}{m_1}\right)}\right) \tag{14}$$

Substitution of Eq. (14) into Eq. (12) and solving for the variable $u_1$, yields:

$$u_1 = \pm\sqrt{\frac{(1-\beta)}{(\alpha-\beta)k_{11}}}. \tag{15}$$

The ellipses intersect when $u_1$ and $u_2$ are members of the set of real numbers. The coordinate $u_1$ is a real number when the following conditions are satisfied: $0 \le \beta \le 1$, $k_{11} > 0$, and $\alpha > \beta$. The first two prerequisites are fulfilled by the definitions of $\beta$ and $k_{11}$. In order to determine what is occurring with the coordinate $u_2$, the restriction $\alpha > \beta$ is assumed. Two terms in Eq. (14) determine if $u_2$ is a real number. These terms are given by

$$2\beta k_{12}u_1 \tag{16}$$

and

$$\sqrt{-\beta\left(\frac{-\beta k_{12}^2u_1^2m_1 + k_{22}u_1^2\alpha k_{11}m_1 + u_1^2\beta m_2k_{22}^2 - k_{22}m_1}{m_1}\right)}. \tag{17}$$

When $u_1$ is a real number, the expression under the radical must always be greater than or equal to zero. Substitution of Eq. (15) into Expression (17) and performing some algebra yields

$$k_{12}^2 - \frac{m_2}{m_1}k_{22}^2 \ge \frac{(1-\alpha)}{(1-\beta)}k_{22}k_{11}. \tag{18}$$

Equation (18) can be transformed by substituting the definitions of $k_{11}$, $k_{12}$, and $k_{22}$ to an expression with of a single damage parameter as

$$\alpha \geq 1 \tag{19}$$

Because $\alpha$ is defined only for the range of $0 \leq \alpha \leq 1$, Eq. (19) is only satisfied for the case of $\alpha = 1$. When this occurs the expression under the radical is zero and $u_2$ will depend only upon the value of $u_1$. This restriction, produces two solutions that satisfy the condition required for the intersection of the ellipses. A similar situation occurs when $\beta = 1$.

When the restriction of $\alpha > \beta$ is not satisfied two boundaries corresponding to an undamaged and damaged state never intersect. If the restriction is enforced, the curves are tangent to each other only for two particular cases, $\alpha = 1$ or $\beta = 1$. These points of tangency are in turn defined by equations that are independent of the damage parameters. The damage state is the convex hull of the undamaged state. This will always be true for any two degree of freedom system model represented by Fig. 2. Because of this fact, the natural frequencies of the structures can be used to represent each particular ellipse.

### 4.1 Example

This example will refer to the system represented in Fig. 2. The values of the first and second mass, $m_1$ and $m_2$, are assumed to be 1 and 4 kilograms respectively. The springs will have the following initial values of $k_1 = 1\frac{\mathrm{N}}{\mathrm{m}}$ and $k_2 = 2\frac{\mathrm{N}}{\mathrm{m}}$. Two damage scenarios will be investigated. These will correspond to a reduction in the stiffness of $k_1$ and $k_2$, by 25% and 50%. Substitution into Eq. (11) establishes an equation which may be used to represent various ellipses which are shown in Fig 3.

$$(1 + 2\beta)u_1^2 - 2\beta u_1 u_2 + 0.5\beta u_2^2 = 1. \tag{20}$$

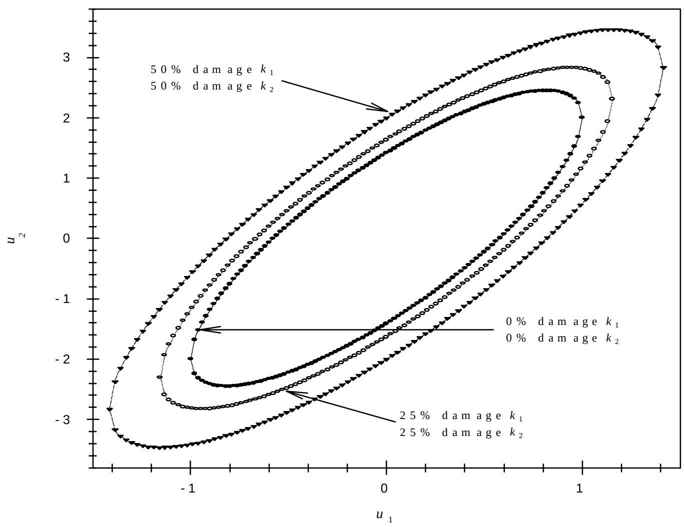


Figure 3: Geometric Interpretation of the Eigenvalue Problem given in Example No. III.

By examining Fig. 3, it can easily be seen that damage occurring in the structure is the convex hull of the condition existing before. It should also be noted that each ellipse in Fig. 3 is unique and represents a particular condition of the system.

## 5. Experimental Verification

An experimental set up consisting of a 6 ft. aluminum beam mounted on top of a VTS VG100-B shaker powered by an Techron 5507 power supply amplifer was used to model a cantilever beam. The beam was mounted with an adhesive to a Kristler 8638B5 accelerometer connected to the shaker. Two Bruel & Kjaer 8309 accelerometers were placed six inches away from the ends of the beam. The signal conditioning was performed with a Kristler 5118A2 power supply coupler and an B&K 2635 Charge type amplifer. These signals were read with an *HP* 35660 *A* dynamic signal analyzer to produce the frequency response function of the structure. The dynamic signal analyzer produced a random signal which was sent to the power supply amplifer to excite the structure. This experimental set up is shown in Fig. 4.

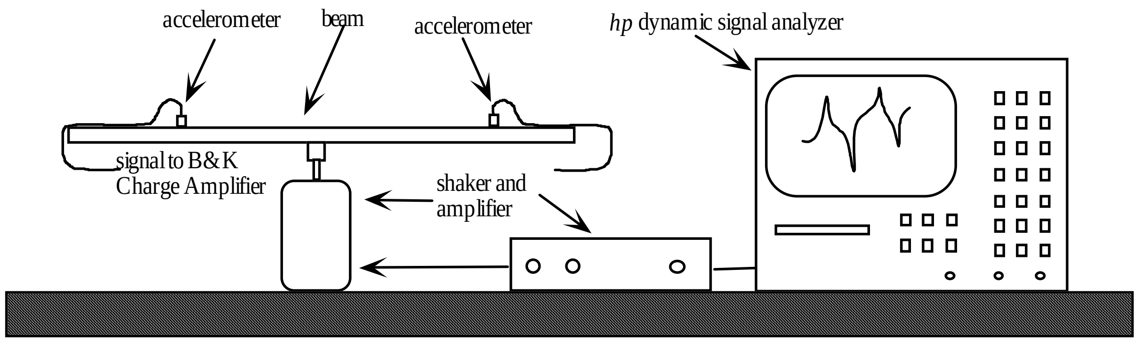


Figure 4: The Cantilever Mounting System used in the Experiment

Damage scenarios were simulated using a two beam finite element model. The results were placed into a spread sheet. Because each eigenvalue problem representation is unique, a norm using the natural frequencies was used to determine the damage location. The beam was damaged at a location of four inches from the cantilever with a slot depth of twenty percent of the beam's height. The structure was then vibrated and the first two natural frequencies were recorded. The beam was removed and another damage state was placed at a new location. This new slot was located sixteen inches from the cantilever end and had a corresponding depth of ten percent. Vibration was repeated and the first two natural frequencies were taken. Damage was augmented at the sixteen inch location to a new slot depth of twenty percent. The vibration experiment was then performed one more time. Figure 5 shows the frequency response function for one of the above mentioned beam conditions and Table 1 shows a typical spread sheet result.

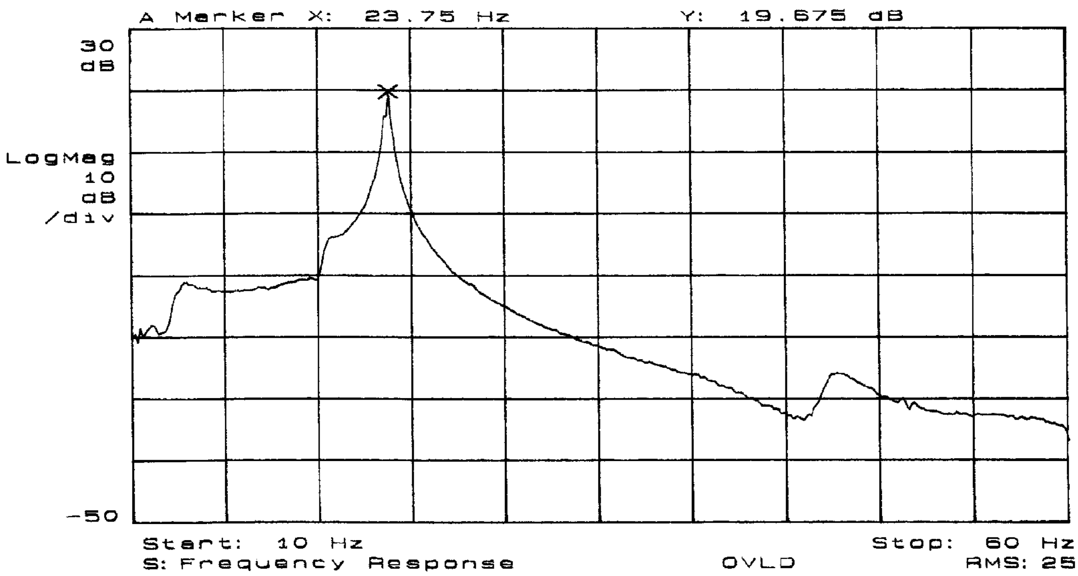

Figure 5: The Frequency Response of a Damaged Cantilever Beam Around the First Natural frequency. ( 20% damage, 4 inches from cantilever)

Table 1: Spread Sheet Results for Damage Located 4 Inches from the Cantilever:

| Damage 20% Deep; Location 4 inches; No Damage in Element II (Reponse in Hz) | | | | | | | |
|---|---|---|---|---|---|---|---|
| Damage in Element No. I | | | | | | | |
| | 0% | 10% | 20% | 30% | 35% | 40% | Actual Response |
| $1^{st}$ natural frequency | 26.89 | 25.57 | 24.17 | 22.66 | 21.86 | 21.03 | 23.75 |
| $2^{nd}$natural frequency | 170.08 | 165.17 | 159.57 | 153.17 | 149.58 | 145.68 | 151 |
| norm | 19.33 | 14.28 | 8.58 | 2.43 | 2.36 | 5.98 | |
| | | | | | | | |
| Damage 20% Deep; Location 4 inches; No Damage in Element I (Reponse in Hz) | | | | | | | |
| Damage in Element No. II | | | | | | | |
| | 0% | 10% | 20% | 30% | 35% | 40% | Actual Response |
| $1^{st}$ natural frequency | 26.89 | 26.82 | 26.73 | 26.61 | 26.54 | 26.46 | 23.75 |
| $2^{nd}$natural frequency | 170.08 | 165.93 | 161.17 | 155.64 | 152.53 | 149.15 | 151 |
| norm | 19.33 | 15.24 | 10.60 | 5.45 | 3.18 | 3.28 | |

### 5.1 Discussion of the Results

The geometric interpretation of the egienvalue problem is a convex set. Damage has the effect of producing a new convex hull of the exsiting mathematical model. This property lets the damage detection method be based upon the closest response of a mathematical model to the actual response of the structure. The example and experimental verification was limited to a very simple model with few possible damage senarios. Even though noise and experimental error were not considered in the analysis, the method performed adequately. The method predicted correctly the damage location (either on the left side or the right side of the beam) for the three cases studied. It should be noted that even though the method performed adequatly, its accuracy is dependant upon the model used.

## 6. Conclusions

Many damage detection schemes use modal parameters such as natural frequency and mode shapes to determine changes in structures. However, none have examined the changes in the geometric representation of the eigenvalue problem which is used in many of the above mentioned schemes. The above examples developed a model for an arbitrary two degree of freedom system shown in Fig. 2. The geometric interpretation of the eigenvalue problem was shown to be a convex set. As damage occurred in the structure, the new geometric interpretation becomes the convex hull of the previous convex set. These results reveal some very important properties about damage detection schemes. One obvious fact is that every new damage scenario will produce unique results in the natural frequencies and in the eigenvectors. This means of examination on how damage effects the parameter changes can reveal a qualitative insight on any detection scheme proposed. Future work in examining the geometric interpretation other solution schemes in engineering might reveal their behavior to parameter changes.

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