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VIBRO-ACOUSTIC MODELING IN INFINITE FLUID MEDIUM USING THE FINITE ELEMENT METHOD AND DtN MAPPING

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Abstract. *The Dirichlet-to-Neumann (DtN) Finite Element Method is a combined numerical-analytic method for boundary value problems in infinite domains. This paper presents a two-dimensional DtN formulation to solve radiation problems in time-harmonic vibro-acoustics systems. The computational aspect related to the DtN Finite Element code implementations are discussed. In this work, the DtN Mapping is applied to solve a coupled fluid-structure problem. The method is used to analyse the acoustic radiation of the free-free plate in infinite air medium. The numerical simulations are in good agreement with the experimental results.*

Key-words: *DtN (Dirichlet-to-Neumann) Mapping, Finite Element Method, Infinite Fluid Medium*

1. INTRODUCTION

The phenomena of interaction between a structure and a fluid are encountered in several domains, for example, in the noise radiated from vehicles, such as: car, buses, and trucks. The simulation of the acoustic behaviour of these systems, it can be useful, when used in the initial stages of the project, to avoid the vibro-acoustic coupling and problems of sound pollution Zavala & Pavanello (1998b).

The Finite Element Method is one of the most versatile methods to model Vibro-acoustic systems. This method has advantages such as: general codes are available, the coupled problems are solved in the direct form and a simple and intuitive simulation. Several approaches using finite element method were developed to model unbounded acoustic domains. These methods could be divided in two categories: special boundary conditions methods and infinite element techniques.

In late 1970's, Bettes and Zienkiewicz has developed the theory of infinite elements Bettes (1992). In this technique a layer of infinite elements is placed around a discretized finite domain with finite elements. Zienkiewicz [3] presented a summary of infinite elements.

In order to solve unbounded domains problems Keller & Givolli (1989) developed a so-called Dirichlet-to-Neumann boundary condition. This approach is an exact and non-reflective method. The DtN Mapping is a non-local boundary condition derived from the exact analytical solution. Givoli (1992) makes an overview about this kind of method and use DtN Mapping in other problems such as aeroelasticity, non-linearity, Laplace Problems, and others. Givoli (1992) presents a comparison of the DtN approach with other special boundary conditions methods, showing the advantages of the DtN Mapping. Zavala & Pavanello (1998b) and Zavala (1999) presents one application of the numerical expansion of the results to the exterior region (not discretized) using the analytical solution.

This paper is concerned with solving the coupled radiation problem, in the low frequency domain, by using the finite element method and the DtN boundary condition. This paper is organised as follows. In section 2, the governing partial differential equations for time-harmonic structural acoustic with radiation boundary condition are summarised and the DtN formulation is introduced. In section 3, the semi-discretized finite element form of the problem is presented. In section 4, the computational aspects of the method are briefly discussed. In section 5, the example of fluid-structure coupling in unbounded fluid medium is solved. Some conclusions are presented in section 6.

2. THE RADIATION PROBLEM

The governing equation of motion including the structural dynamics, the acoustics, and their coupling, are recalled. The acoustic radiation problem, Fig. 1, can be represented by the Helmholtz equation in velocity potential, ψ, in an infinite domain, $\mathcal{R}$, with internal boundary, Γ, where $\Gamma = \Gamma_g \cup \Gamma_h$, as follow (Keller & Givoli 1989):

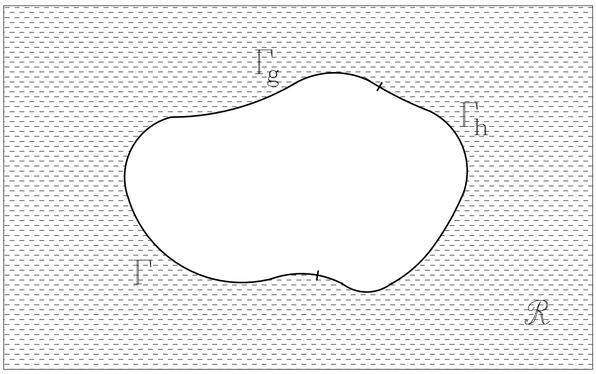


Figura 1: Infinite Fluid Domain with Internal Boundary

$$\nabla^2\psi + k^2\psi + f = 0 \qquad in\ \mathcal{R} \tag{1}$$

$$\psi = g \qquad in\ \Gamma_g \tag{2}$$

$$\psi_{,n} = ikh \qquad in\ \Gamma_h \tag{3}$$

$$\lim_{r\to\infty} r^{(d-1)/2}(\psi_{,r} - ik\psi) = 0 \tag{4}$$

where ∇^2 is the Laplacian operator; k is the wave number ($k = \omega/c$); f is the contribution of acoustic sources; $\psi_{,n}$ is the differentiation in exterior normal direction $\vec{n}$; i is the imaginary unit; g and h are known functions in Dirichlet and Neumann boundary conditions respectively; r is the distance from de origin and d is the spatial dimension. Boundary Conditions like Free Boundaries, Rigid Walls, Fluid-structure interface and others can be represented by Eq. (2,3). Equation (4) is the Sommerfeld Radiation Boundary Condition, and represents the infinite external fluid medium. The solution of the problem Eq. (1 - 4) can be found analytically for some simple and classic problems with regular geometry, but the analytical solutions for more complex geometry are practically impossible.

2.1. The infinite fluid medium approach with DtN mapping

To obtain the approximate solution of the problem Eq. (1 - 4), an equivalent problem consisting of to separated domains by an artificial boundary can be resolved (Keller & Givoli 1989). The system is divided in an infinite external domain, $\mathcal{D}$, where the solution of the homogenous Helmholtz Equation is required, and an internal finite domain, Ω, where the solution of the non-homogenous Helmholtz equation is required. A simple representation of the problem can be seen in the Fig. 2.

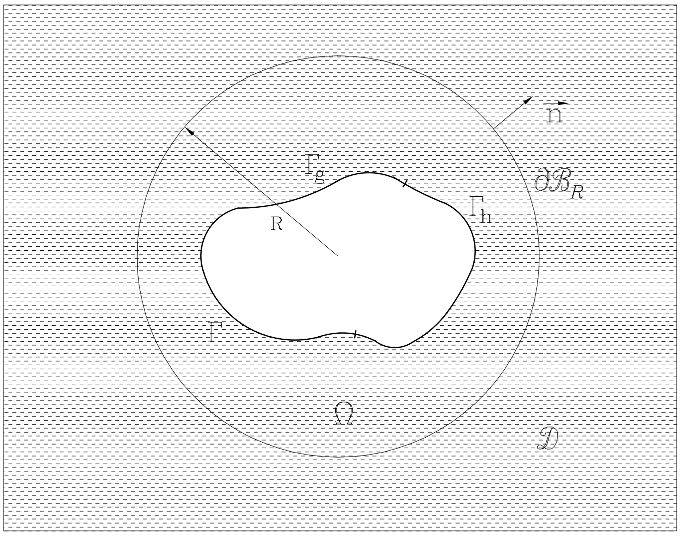


Figura 2: Equivalent Problem

The Dirichlet problem defined in $\mathcal{D}$, can be represented by:

$$\nabla^2\psi + k^2\psi = 0 \qquad in\ \mathcal{D} \tag{5}$$

$$\psi = \psi(R,\theta) \qquad in\ \partial\mathcal{B}_R \tag{6}$$

$$\lim_{r\to\infty} r^{(d-1)/2}(\psi_{,r} - ik\psi) = 0 \tag{7}$$

In two dimensions ($d = 2$), the artificial boundary is a circle with radius R, and the solution of Eq.(5 - 7) is as follows (Morse & Fesbach 1953):

$$\psi(r,\theta) = \frac{1}{\pi}{\sum_{n=0}^{\infty}}' \int_0^{2\pi} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \cos n(\theta-\theta_H)\, \psi(R,\theta_H) d\theta_H \tag{8}$$

where r and θ are the radius and the angle of an exterior point; n is the number of harmonic terms of the solution; $H_n^{(1)}$ is the Hankel function of first type; θ_H is the angle used to the integration in the artificial boundary. The prime after the sum indicates that a factor of $1/2$ multiplies the term with $n = 0$.

On $\partial\mathcal{B}_R$ the following boundary condition can be imposed

$$\psi_{,r} = M\psi \qquad\qquad in\ \partial\mathcal{B}_R \tag{9}$$

where M is an operator called the Dirichelt to Neumann (DtN) map. Differentiating the equation (8) with respect to r in the artificial boundary $\partial\mathcal{B}_R$, yields

$$\psi_{,r}(R,\theta) = \sum_{n=0}^{\infty}{}' \frac{k}{\pi}\frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)}\int_0^{2\pi}\cos n(\theta-\theta_H)\psi(R,\theta_H)d\theta_H \tag{10}$$

where the prime symbols after a function means differentiation with respect to its argument. Thus, the DtN Operator could be expressed as:

$$M\psi = \sum_{n=0}^{\infty}{}' \alpha_n \int_0^{2\pi}\cos n(\theta-\theta_H)\psi(R,\theta_H)d\theta_H \tag{11}$$

where

$$\alpha_n = -\frac{k}{\pi}\frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)} \tag{12}$$

Finally, the DtN Problem can be represented as:

$$\begin{aligned}
\nabla^2\psi + k^2\psi + f &= 0 && in\ \Omega && (13)\\
\psi &= g && in\ \Gamma_g && (14)\\
\psi_{,n} &= ikh && in\ \Gamma_h && (15)\\
\psi_{,n} &= -M\psi && in\ \partial\mathcal{B}_R && (16)
\end{aligned}$$

For the finite domain Ω, the Finite Element Method can be applied directly (Keller & Givoli 1989), and the classical fluid structure model can be used. For a continuous solid, the elasticity relationships, without dissipative terms are:

$$\sigma_{ij,j}(u) - \rho_s\ddot{u}_i = 0 \qquad\qquad in\ \Omega_s \tag{17}$$

where σ_{ij} is the stress tensor; ρ_s is the mass structural density and u_i are the components of the structural displacements. The local coupling conditions at the fluid-structure interfaces are:

$$\begin{aligned}
\dot{\psi}_{,n} &= \dot{u}_n && (18)\\
\sigma_{ij}(u)n_i &= -\rho_f\dot{\psi}_{,n} && (19)
\end{aligned}$$

Equations (13) to (19) are the governing equations of the coupled radiation problem.

3. FEM APROXIMATIONS OF DtN MAPPING

The DtN Mapping is a natural boundary condition which can be included in the weak form of the fluid problem as follows (Zienkiewicz & Taylor 1991):

$$\int_{\Gamma} w\frac{\partial\psi}{\partial n}d\Gamma = \int_{\Gamma_g} w\frac{\partial\psi}{\partial n}d\Gamma + \int_{\Gamma_h} w\frac{\partial\psi}{\partial n}d\Gamma + \int_{\partial\mathcal{B}_R} w\frac{\partial\psi}{\partial n}d\Gamma \tag{20}$$

where w is the weight function. Using a classical FEM approximation, combining Eq.(16) and Eq.(20), the integral equation in $\partial\mathcal{B}_R$ becomes

$$\int_{\partial\mathcal{B}_R} w\frac{\partial\psi}{\partial n}d\Gamma = -\int_{\partial\mathcal{B}_R} wM\psi d\Gamma \tag{21}$$

Using Galerkin method and a classical finite element approximation, the matricial form of the DtN kernel can be expressed by:

$$-\left[\int_{\partial\mathcal{B}_R} N_i M N_j d\Gamma\right]\psi_j = [D]\{\psi_{\partial\mathcal{B}_R}\} \tag{22}$$

To calculate the matrix $[D]$, considering the two-dimensional problem in infinite medium, the expression of the DtN operator M can be rewritten for the artificial circular boundary, Eq. (11), with variables in separately arrange,

$$M\psi = \sum_{n=0}^{\infty}{}' \alpha_n \int_0^{2\pi} \left(\cos n\theta \cos n\theta_H + sen\ n\theta\ sen\ n\theta_H\right)\psi(R,\theta_H)d\theta_H \tag{23}$$

or,

$$M\psi = \sum_{n=0}^{\infty}{}' \alpha_n \left(\cos n\theta \int_0^{2\pi} \cos n\theta_H \psi(R,\theta_H)d\theta_H + sen\ n\theta \int_0^{2\pi} sen\ n\theta_H \psi(R,\theta_H)d\theta_H\right) \tag{24}$$

Now, substituting the expression of the DtN operator in Eq.(22),

$$\begin{aligned} D_{ij} \ = \ & -\sum_{n=0}^{\infty}{}' \alpha_n \left[\left(\int_{\partial\mathcal{B}_R} N_i \cos n\theta\ d\Gamma\right)\left(\int_0^{2\pi} N_j \cos n\theta_H\ d\theta_H\right)\right. \\ & \left. + \left(\int_{\partial\mathcal{B}_R} N_i\ sen\ n\theta\ d\Gamma\right)\left(\int_0^{2\pi} N_j\ sen\ n\theta_H\ d\theta_H\right)\right] \end{aligned} \tag{25}$$

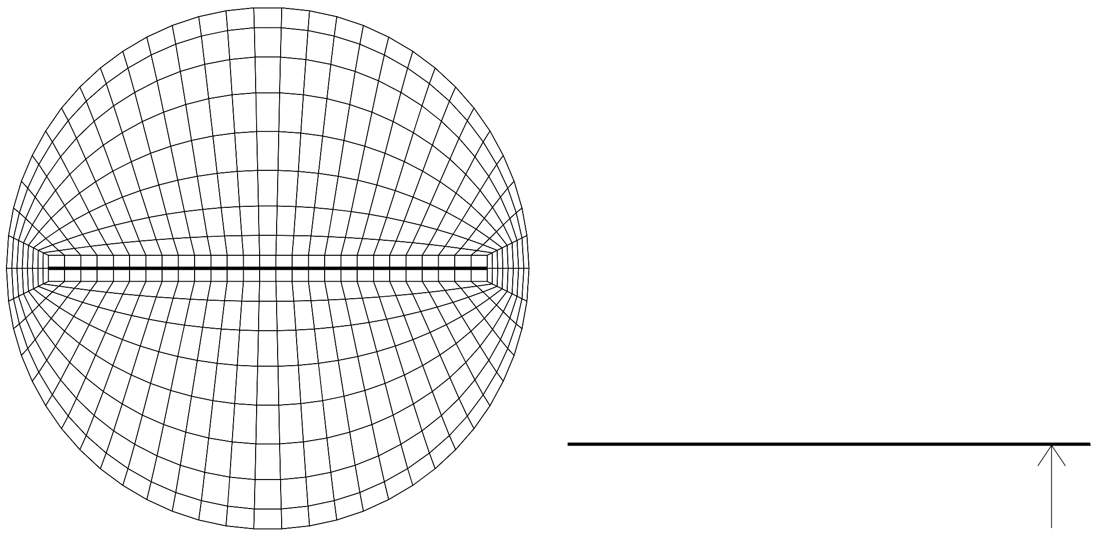
To obtain the matrix $[D]$, $2 \times n_{\mathcal{B}}$ integrals have to be evaluated for each term of the sequence in n, where $n_{\mathcal{B}}$ is the number of nodes in the artificial boundary. The influences of the radius of the artificial boundary, the number of terms in the DtN Kernel evaluation, and the mesh refinement required for good spatial and frequency resolution, are discussed in Zavala(1999), Zavala & Pavanello (1998a) and Givoli(1992).

4. COMPUTATIONAL ASPECTS IN THE INCLUSION OF THE DtN MAPPING

The discretized equation of the acoustic system in infinite fluid medium can represented as:

$$[S]\{\ddot{\psi}\} + ([H] + [D])\,\{\psi\} = \{F\} \tag{26}$$

where $[S]$ is the inertial matrix; $[H]$ is the volumetric matrix; $\{F\}$ is the acoustic sources vector; $\{\psi\}$ is the nodal variables vector, and $\ddot{}$ means the second time differentiation. The solution of the equation (25) is non-local, and the evaluation of the variable at each position depends on all others variables in the artificial boundary, $\partial\mathcal{B}_R$. This results in coupling all variables in the boundary. The inclusion of all terms of the DtN Mapping is directly done in the global matrix of the system. This spoils the sparseness of resulting matrix $[H] + [D]$. However a bandwidth and profile reduction method can be used to re-establish a sparse finite element matrix.



(a) (b)

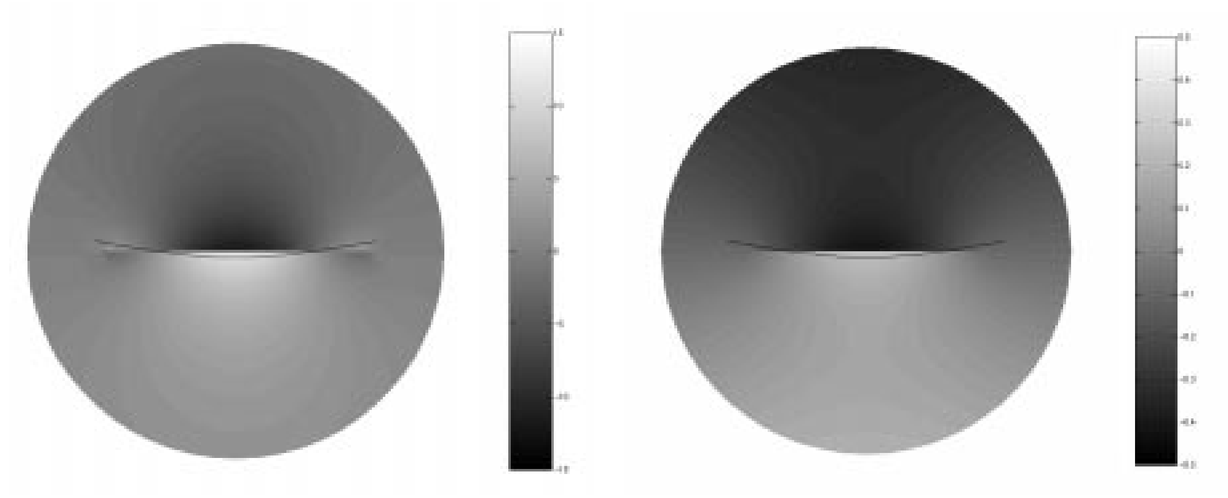
Figura 3: (a) Two-dimensional model mesh, (b) excitation point.

The computational cost associated to the evaluation of the Bessel functions is small. This functions are just evaluated one time for each kR, and a small number of harmonic terms n is required. The biggest cost in DtN Kernel calculations is associated to evaluation of the integrals of the trigonometric functions. This cost is minimised if the integrals are obtained explicitly, Zavala (1999).

Harari & Hughes (1992), shown that the computational costs associated to the resolution of acoustic problems in infinite fluid medium using the FEM with DtN Mapping compared to the Boundary Element Method (BEM) have a general advantage.

5. EXAMPLE OF FLUID-STRUCTURE COUPLING IN INFINITE FLUID MEDIUM

To illustrate the ability of the DtN Method for a real system, the radiation problem of a free-free plate is dealt with.

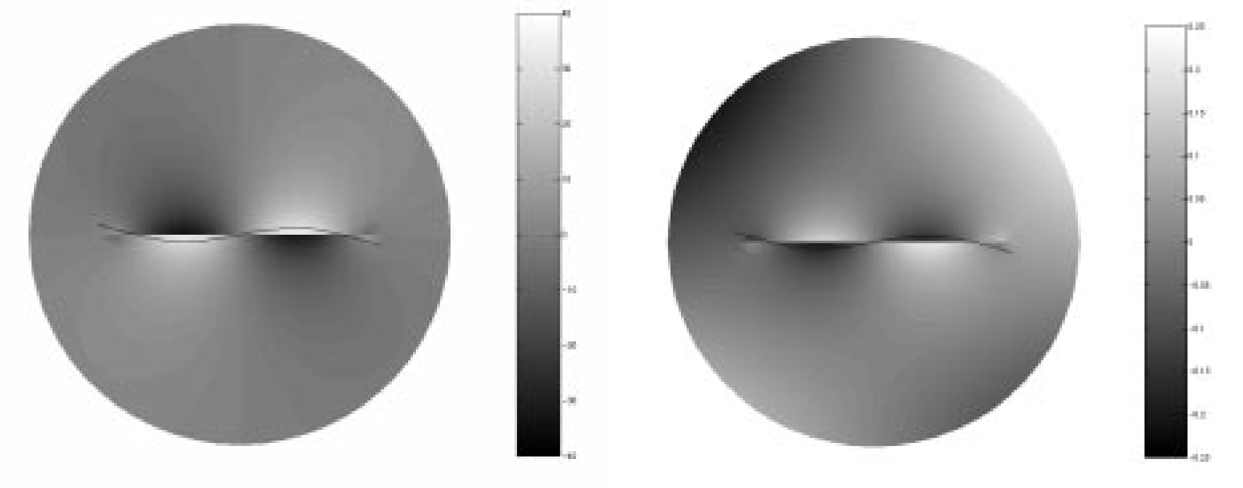


(a) (b)

Figura 4: First Coupled Operational Mode . (a) Real Part (b) Imaginary Part.

The plate dimensions are $672 \times 217 \times 3,2$ mm. The plate material is aluminium, with density of 2700 $\mathrm{Kg/m^3}$, and elastic modulus of $7,1 \times 10^{10}$ N/m.

A two-dimensional model consisting of 27 Euler-Bernouilli beam elements, with geometric properties of: area of $0,0032$ $\mathrm{m^2}$, and Moment of Inertia of $2,7307 \times 10^{-9}$ $\mathrm{m^4}$ is used to model the central section of the plate. The fluid domain were discretized with 518 quadrilateral bi-linear elements, with properties of (air): density of $1,2$ $\mathrm{Kg/m^3}$; and sound velocity of 342 m/s.

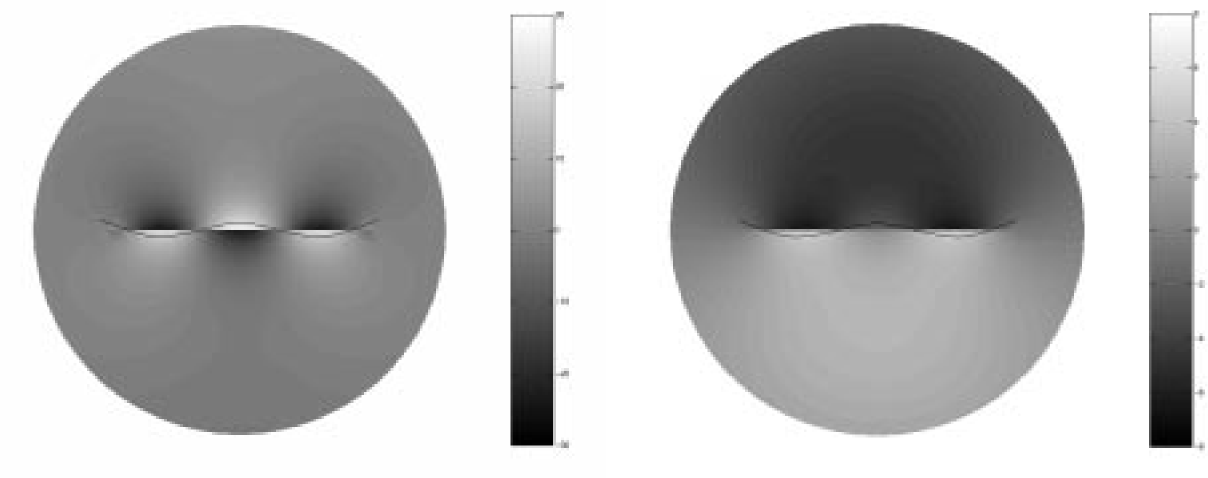


(a) (b)

Figura 5: Second Coupled Operational Mode. (a) Real Part (b) Imaginary Part.

The adopted finite element mesh, with artificial boundary radius of $R = 0.50$ m, can be seen in Figure 3. The excitation points are shows in Figure 3. The results are obtained using $n = 10$ DtN kernel terms.

We present in Figures 4 to 6 the theoretical fluid and structural operational mode for the first three flexural modes of the structure, where the amplitudes of the structure displacement were augmented to better visualisation of the shapes. The pressure response results are in Pa ($\mathrm{N/m^2}$). The frequencies of the operational modes are $37.5Hz$,



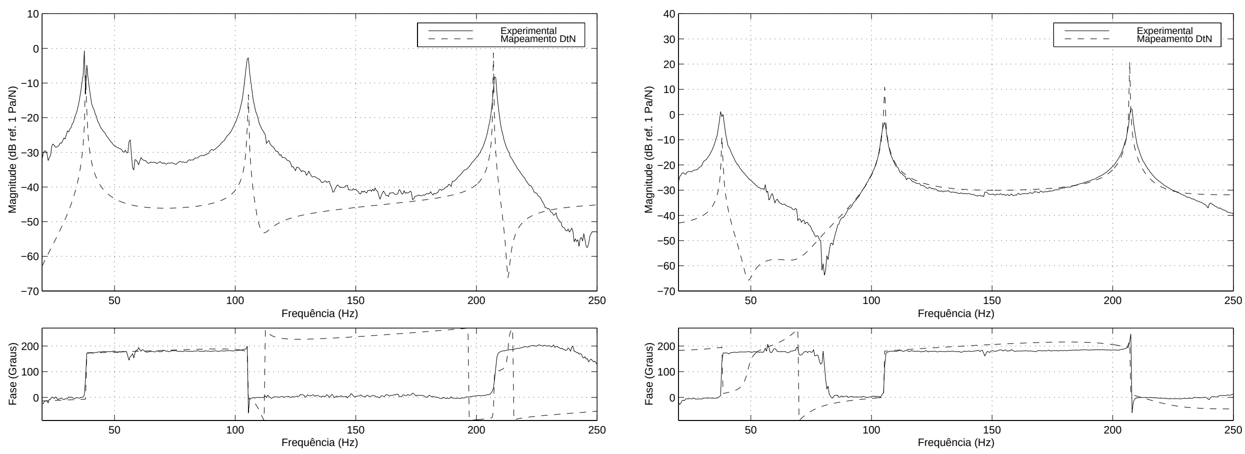
(a) (b)

Figura 6: Third Coupled Operational Mode. (a) Real Part (b) Imaginary Part.

104.3$Hz$and 205.5Hz.

The measurements of the pressure field were done in an anechoic field. The measurement points, in a total of 28, are distant 2 cm of the plate, in the direction of the longitudinal central line, equally spaced. The complete description of the experiment can be found in COLINAS (1999).

The numerical and experimental frequency response functions between the pressure and the excitation force were extracted. The comparison of the experimental and numerical results can be seen in Figures 7 to 9.

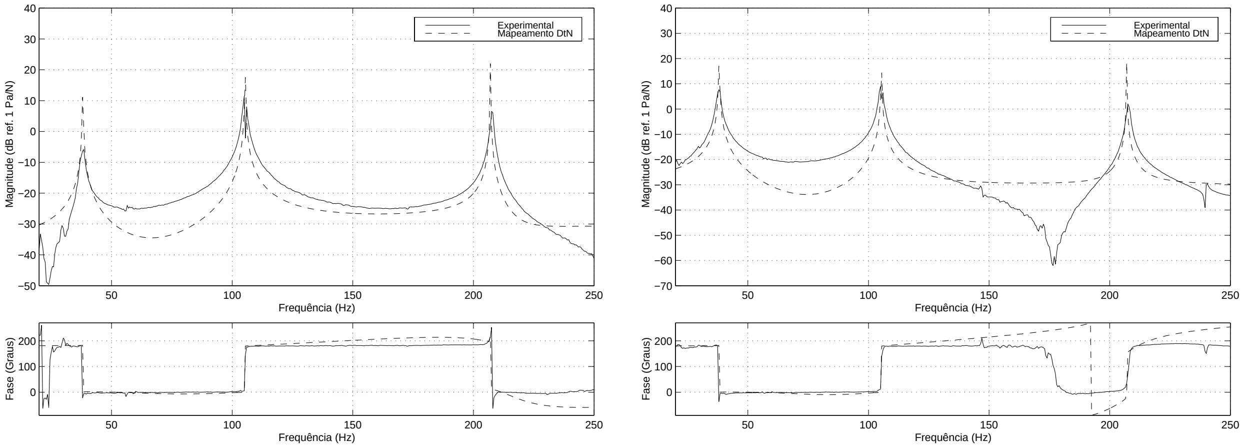


(a) (b)

Figura 7: FRF Comparison. (a) $x = 0.0$ m (b) $x = 0.100$ m.

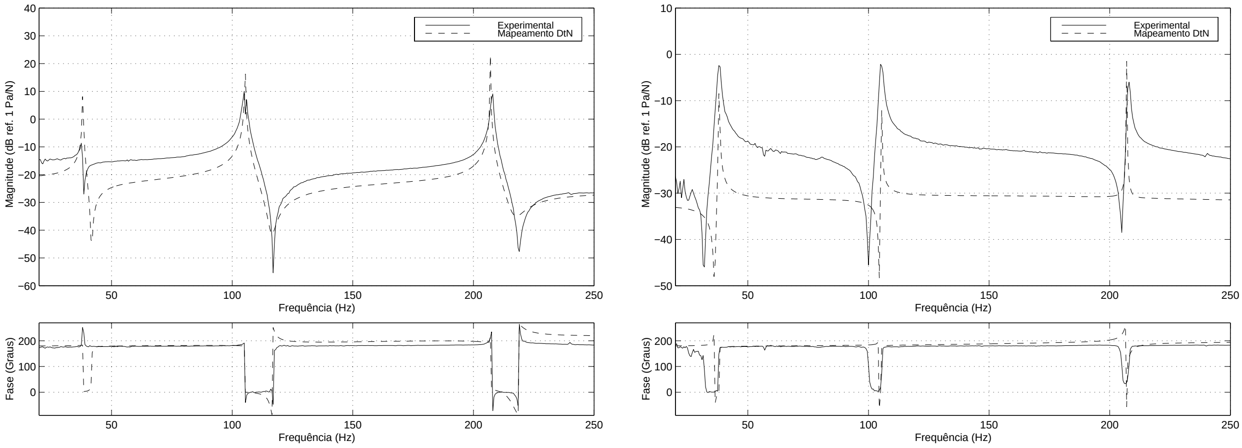
In Figures 10 (a), (b) and (c), are made a comparison of real parts of the fluid response along the entire plate for all measurement points.

In view of the results presented here, the capability of the DtN Method to predict the coupled vibro-acoustic operational modes is quite satisfactory.



(a) (b)

Figura 8: FRF Comparison. (a) $x = 0.174$ m (b) $x = 0.274$ m.



(a) (b)

Figura 9: FRF Comparison. (a) $x = 0.523$ m (b) $x = 0.672$ m.

6. CONCLUSIONS

A method for the computation of frequency response function and operational modes of coupled unbounded vibro-acoustic systems is described. The application to free-free plate in infinite air medium, with experimental verification of computed results has demonstrated its efficiency.

The results obtained with DtN Method, used with FEM, showed satisfactory accordance with the measurement results for the example presented.

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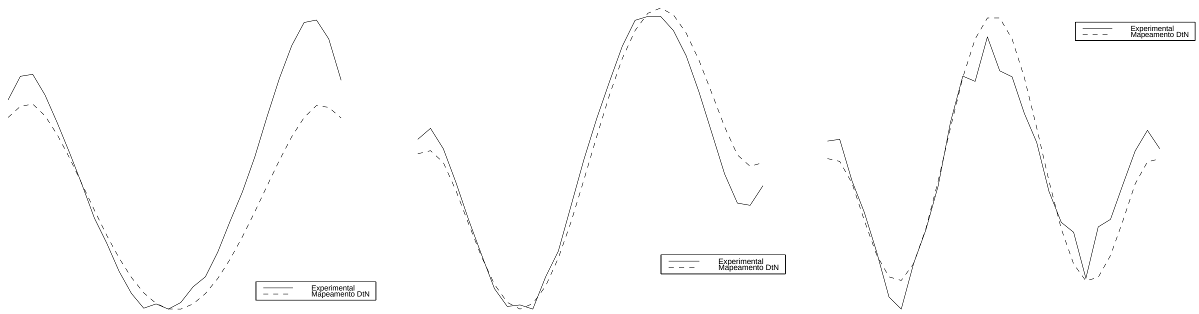

(a) (b) (c)

Figura 10: Pressure Modes Comparison. (a) First Mode (b) Second Mode (c) Third Mode.

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