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HYPERBOLIC DESCRIPTION OF CONTAMINATED FLOW THROUGH AN UNSATURATED WELLBORE

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Abstract. *This work studies the flow of a mixture of two fluids – a Newtonian fluid and a pollutant – through a rigid cylindrical shell porous matrix. Aiming to build a preliminary local model for the flow of a Newtonian fluid containing a pollutant through a wellbore, a mixture theory approach is employed. The mixture consists of four overlapping continuous constituents: one solid (porous medium), one liquid (Newtonian fluid), the pollutant (solid, liquid or gas) and an inert gas included to account for the compressibility of the mixture as a whole. Assuming the flow on radial direction only, a set of three nonlinear partial differential equations describes the problem. Combining Glimm's scheme with an operator splitting technique to account for the non-homogeneous part of the hyperbolic operator, the resulting nonlinear hyperbolic system is numerically approximated. Representative results illustrating the numerical methodology are presented.*

Keywords. Polluted flow, porous medium, Glimm's scheme, operator splitting technique.

1. Introduction

Transport phenomena in porous media are related to problems that impact the energy self-sufficiency and the environmental state. Important examples of the latter are groundwater flows and pollutant contamination of soils. Unsaturated porous media, according to Tien and Vafai (1990), have been studied since the 1920s. The drying phenomenon was simulated supposing that the fluid motion through the porous medium was caused by diffusion only (the balance of linear momentum was substituted by the diffusion equation). In other studies, the influence of capillary forces (surface tension) in the modeling of liquid motion has also been considered.

Most of the works concerning transport phenomena in porous media employ a volume averaging technique – describing concentration and velocity components as volumetric averages in order that a classical continuum mechanics context may be used to study the momentum transport. A comprehensive review, comparing different models for complex problems of transport in porous media employing this approach is found in Alazmi and Vafai (2000).

A distinct approach is used in this work – a mixture theory one, in order to study the dynamics of the flow of a mixture of two fluids – a Newtonian fluid and a dispersed pollutant – through a rigid cylindrical shell porous matrix. A preliminary local description for the flow of these two constituents through a wellbore is performed by employing a mixture theory approach. A mixture of four overlapping continuous constituents is considered: a solid (porous medium), a liquid (representing the Newtonian fluid), the pollutant – which may be solid (very small particles), liquid or gas and, finally, an inert gas included to account for the compressibility of the mixture as a whole.

Assuming an isothermal flow, the mixture theory requires momentum sources to account for the mechanical coupling among the constituents. Besides the momentum source – usually called a Darcian term – the cylindrical shell geometry gives rise to other source terms, which would be absent in a rectangular geometry, for instance. Supposing the porous matrix rigid and at rest and since the gas constituent is present for allowing changes in the liquid fraction (or liquid concentration) only – so that, no equation is required to describe its behavior, it suffices to solve the momentum and mass balance equations for the constituents representing the Newtonian fluid and the pollutant. At this point an important hypothesis is considered – the amount of pollutant present, at any time instant is so small, compared with the Newtonian liquid, that it is convenient to consider a pseudo-constituent to represent the mixture of the Newtonian fluid and the pollutant. As a consequence, the motion problem will be reduced to solving mass and linear momentum balance equations for this pseudo-constituent combined with the pollutant mass balance.

The radial flow of the above described mixture through a porous cylindrical shell, considering a strongly advective process, consists of a nonlinear hyperbolic system of three equations whose approximation is performed by combining Glimm's scheme – specially designed to deal with problems that may give rise to shock waves and an operator splitting technique – an effective tool to account for the non-homogeneous part of the differential equations.

2. Mechanical model

This work studies the flow of a polluted fluid through a porous medium, the polluted fluid being composed by a Newtonian fluid and a very small amount of pollutant – this latter consisting of very small solid particles, liquid, or gas, dispersed in the Newtonian fluid. The small quantity of pollutant has motivated an alternative mixture theory approach – namely considering the mixture of a Newtonian fluid and a pollutant as a pseudo-constituent of a mixture, which, from now on will be conveniently denoted as fluid constituent. The above-mentioned assumption allows considering, in turn, this so-called pseudo-constituent as a single constituent of a mixture, concerning mass and momentum balance equations.

As a consequence, the motion of a polluted fluid through a porous medium will be described by mass and momentum balance equations for the pseudo-constituent combined with the mass balance for the constituent representing the pollutant. The motion of a mixture of isothermal constituents under a mixture theory approach is usually described by mass and momentum equations for every constituent of the mixture and for the mixture as a whole. This work assumes a chemically non reacting continuous mixture of a rigid solid constituent at rest, a pseudo-constituent (representing the polluted fluid – actually a very small quantity of pollutant dispersed in the Newtonian fluid) – a liquid, denoted as fluid constituent and an inert gas playing the role of an additional constituent included to account for the mixture compressibility. Under all these hypotheses, it suffices to solve mass and momentum balance equations for this so-called fluid constituent as well as the mass balance for the constituent representing the pollutant. The fluid constituent mass balance is given by (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$\frac{\partial \rho_F}{\partial t} + \nabla \cdot (\rho_F \mathbf{v}_F) = 0 \tag{1}$$

where ρ_F stands for the fluid constituent (pseudo-constituent) mass density – representing the local ratio between the fluid constituent mass and the corresponding volume of mixture and $\mathbf{v}_F$ is the fluid constituent velocity in the mixture.

The balance of linear momentum for the fluid constituent is given by (Atkin and Craine, 1976; Rajagopal and Tao, 1995)

$$\rho_F \left[\frac{\partial \mathbf{v}_F}{\partial t} + (\nabla \mathbf{v}_F)\mathbf{v}_F \right] = \nabla \cdot \mathbf{T}_F + \mathbf{m}_F + \rho_F \mathbf{b}_F \tag{2}$$

in which $\mathbf{T}_F$ represents the partial stress tensor – analogous to Cauchy stress tensor in Continuum Mechanics – associated with the fluid constituent, the body force (per unit mass) is represented by $\mathbf{b}_F$ while $\mathbf{m}_F$ is the momentum supply acting on the fluid constituent due to its interaction with the remaining constituents of the mixture. In other words, the interaction among the pseudo-constituent, the solid constituent and the inert gas. This momentum source is an internal contribution, so the net momentum supply to the mixture due to all the constituents must be zero: $\sum_{i=1}^{n} \mathbf{m}_i = 0$. The balance of angular momentum is satisfied through an adequate choice of $\mathbf{T}_F$. Assuming the partial stress tensor symmetrical, it is automatically fulfilled.

The ratio between the fluid fraction φ and the porous matrix porosity ε is defined as the saturation ψ, so that

$$\psi = \frac{\varphi}{\varepsilon} = \frac{\rho_F}{\varepsilon \rho_f} \qquad 0 < \psi \leq 1 \quad \text{everywhere} \tag{3}$$

in which ρ_f is the actual mass density of the polluted fluid – regarded as a single continuum, in contrast to ρ_F defined as the fluid constituent (the pseudo-constituent) mass density.

According to Williams (1978) and Saldanha da Gama and Sampaio (1987), the following constitutive relation may be considered for the momentum source term – which accounts for the dynamic interaction among the constituents, in a mixture representing an unsaturated flow of an incompressible Newtonian fluid through a porous matrix

$$\mathbf{m}_F = -\alpha \psi^2 \mathbf{v}_F - \beta \psi \nabla \psi \qquad \text{with} \quad \alpha = \frac{\varepsilon^2 \mu_f}{K} \quad \beta = \frac{\varepsilon^2 \mu_f \mathcal{D}}{K} \tag{4}$$

where μ_f represents the fluid viscosity (measured considering a Continuum Mechanics viewpoint), K the porous matrix specific permeability and $\mathcal{D}$ a diffusion coefficient – analogous to the usual mass diffusion coefficient.

An analogy with the stress tensor acting on an incompressible Newtonian fluid within a Continuum Mechanics framework probably led Williams (1978) to consider the partial stress tensor acting on the fluid constituent as being

proportional to the pressure acting on it and to the gradient of its velocity, suggesting a constitutive relation analogous to the usually employed for Cauchy stress tensor with such a behavior. A further simplification has been later proposed by Allen (1986), who concluded that among the three distinct momentum transfer mechanisms in the mixture – namely: shear stresses, interphase tractions and momentum transfer through fluid drag on the porous matrix, the normal fluid stresses were dominant, the shear stresses and interphase tractions being negligible when compared to the fluid drag, so the partial stress tensor may be approximated by the following relation

$$\mathbf{T}_F = -\varepsilon^2 \overline{p} \psi^2 \mathbf{I} \tag{5}$$

where $\overline{p}$ is a pressure (assumed constant while the flow is unsaturated) and $\mathbf{I}$ is the identity tensor.

2.1. Pollutant mass balance

Assuming the mass transfer as caused by an advection-diffusion process of the pollutant in the binary-mixture (representing the Newtonian fluid and the pollutant), the pollutant mass balance is given by

$$\frac{\partial(\rho_F \omega)}{\partial t} + \nabla \cdot (\rho_F \omega \mathbf{v}_F) = \nabla \cdot (\rho_F D \, \nabla \omega) + r \tag{6}$$

in which ρ_F stands for the pseudo-constituent (Newtonian fluid + pollutant) mass density and $\mathbf{v}_F$ for its velocity. The concentration of the pollutant in the binary mixture (representing the Newtonian fluid and the pollutant), ω, is defined as the mass fraction of the pollutant in the binary mixture, being expressed by the following equation $\omega \equiv \rho_{pol} / \rho_F$, in which ρ_{pol} stands for the pollutant mass density regarded as a constituent of the mixture – in such a way that the pollutant actual mass is obtained by its integration over the whole mixture. Besides, D represents the diffusion coefficient of the pollutant in the binary mixture and r the rate of production of the pollutant per unit volume.

In the absence of any chemical reactions, which could alter the quantity of the pollutant, it comes that the production term r is zero. Also, diffusion will be neglected, in a first approach, so that the diffusion coefficient D is zero. This hypothesis stands for a very high advective flow, which is considered in this work. As a consequence of these two assumptions, the right-hand term of equation (6) vanishes.

2.2. Onedimensional phenomenon

In this section a one dimensional approach is used to simulate the flow of a binary mixture – representing a pollutant dispersed in a Newtonian fluid – through a cylindrical shell porous matrix, assuming that all the quantities depend only on the time *t* and on the position *r* and that *v* is the only non vanishing component of the fluid constituent (pseudo-constituent) velocity $\mathbf{v}_F$. Under these assumptions the balance equations (1), (2) and (6) combined with saturation definition – Eq. (3) – and the constitutive relations (4)-(5) give rise to the following nonlinear system

$$\begin{aligned}
&\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial r}(\psi v) = -\frac{\psi v}{r} \\
&\rho_f \varepsilon \left[\psi \frac{\partial v}{\partial t} + \psi v \frac{\partial v}{\partial r} \right] = -\frac{\partial}{\partial r}\left(\varepsilon^2 \psi^2 \overline{p}\right) - \frac{\beta \rho_f^2 \varepsilon^2}{2} \frac{\partial \psi^2}{\partial r} - \alpha \psi^2 v \\
&\frac{\partial}{\partial t}(\psi \omega) + \frac{\partial}{\partial r}(\psi \omega_A v) = -\frac{\psi v \omega}{r}
\end{aligned} \tag{7}$$

A reference pressure p_0 is now defined as $p_0 = \overline{p} + \frac{\beta \rho_f^2}{2}$. Also, the following dimensionless quantities are employed to present Eq. (7) in a dimensionless form:

$$u = v\sqrt{\frac{\rho_f}{\varepsilon p_0}} \qquad \tau = \frac{t}{r_*}\sqrt{\frac{\varepsilon p_0}{\rho_f}} \qquad \gamma = \frac{\alpha r_*}{\rho_f \varepsilon}\sqrt{\frac{\rho_f}{\varepsilon p_0}} \qquad \xi = \frac{r - r_i}{r_e - r_i} \tag{8}$$

in which $r_* = r_e - r_i$ with r_e and r_i standing for the external and internal radii of the cylindrical shell matrix. It is worth mentioning that the external radius r_e will be used as a reference in the numerical simulation only, since the numerical results will be presented in a region from r_i to r_e. Boundary conditions will be imposed solely at the cylindrical shell

internal surface. Equations. (8) and the reference pressure p_0 allow rewriting the nonlinear system of equations (7) in a more convenient form as

$$\begin{aligned} \frac{\partial \psi}{\partial \tau} + \frac{\partial}{\partial \xi}(\psi u) &= -\frac{\psi u}{\xi} \\ \frac{\partial}{\partial \tau}(\psi u) + \frac{\partial}{\partial \xi}(\psi u^2 + \psi^2) &= -\frac{\psi u^2}{\xi} - \gamma \psi^2 u \\ \frac{\partial}{\partial \tau}(\psi \omega) + \frac{\partial}{\partial \xi}(\psi \omega u) &= -\frac{\psi u \omega}{\xi} \end{aligned} \tag{9}$$

3. Numerical approximation

In this section a scheme, developed to simulate nonlinear hyperbolic problems, is employed to obtain numerical approximations for the nonlinear system of partial differential equations described in Eqs. (9). Two main ingredients have been used to achieve this goal: an operator splitting technique together with the Glimm's scheme, successfully employed in the simulation of other nonlinear hyperbolic problems. The procedure consists in a decomposition of the operator in such a way that its merely hyperbolic part is split away from its purely time evolutionary one. Glimm's method, specifically developed to deal with hyperbolic non-linear problems, consists in marching from a time *n* to a time *n*+1 through the solution of the associated Riemann problem for each two consecutive time steps. It is based on a theory whose mathematical formulation has a solid thermodynamic basis, which could be expressed by the entropy condition (Smoller, 1983). Concerning Glimm's method simulation of nonlinear hyperbolic systems, a comprehensive approach is presented in Martins-Costa and Saldanha da Gama (2003) where this method is employed to the simulation of a pollutant transport in the atmosphere.

A wide range of non-linear hyperbolic problems have already been simulated by combining Glimm's scheme and an operator splitting technique among which are the wave propagation in gas pipelines, shock propagation in gas dynamics problems and wave propagation in a damageable elasto-viscoplastic pipe (see Freitas Rachid et al., 1994 and references therein). Other relevant examples that could be quoted are the response of non-linear elastic rods (Saldanha da Gama, 1990) and the isothermal and non-isothermal flow of either ideal or Newtonian fluids through unsaturated porous media – covering most one-dimensional cases of interest (see Martins-Costa and Saldanha da Gama, 2001, and references therein). It is remarkable that the problems addressed in these works, due to their hyperbolic nature, do not require boundary conditions. They are essentially initial value problems (John, 1982).

Glimm's method, which deals with the homogeneous part of the hyperbolic operator represented in Eqs. (9) employs the solution of the associated Riemann problem to march from a time *n* to a time *n*+1. Before using Glimm's scheme for solving Eqs. (9) with appropriated initial data, the solution of the associated Riemann problem must be known. In short, Glimm's method allows building a solution for an initial value problem – namely nonlinear hyperbolic systems subjected to arbitrary initial data, through the solution of a certain number of associated Riemann problems. The arbitrary initial condition given by a function of the position ξ is approximated by piecewise constant functions, known as step functions – with equal width steps. In the sequence a Riemann problem – an initial value problem whose initial condition must be a step function, is to be solved for each two consecutive steps. The main idea behind the method is to appropriately gather the solution of as many Riemann problems as desired to successively march from time $\tau = \tau_n$ to time $\tau_{n+1} = \tau_n + \Delta\tau$.

At this point the following redefinition of variables is performed in order to express the system (9) in a more convenient way:

$$F \equiv \psi, \quad G \equiv \psi u \quad \text{and} \quad H \equiv \psi \omega \tag{10}$$

The first step consists in obtaining an initial approximation for (F, G, H) by advancing $\Delta\tau$ in time through the homogeneous (merely hyperbolic) part of the operator via Glimm's method, using the values of (F, G, H) at time $\tau = \tau_n$ as initial data. The numerical approximation for the solution at time $\tau = \tau_{n+1}$ is then obtained by advancing in time with the same time step $\Delta\tau$ through the purely time evolutionary system. This procedure is repeated until reaching a specified simulation time.

The numerical procedure employed to advance from the time $\tau = \tau_n$ to $\tau = \tau_{n+1}$, considering $F = \hat{F}_n(\xi, \tau)$, $G = \hat{G}_n(\xi, \tau)$ and $H = \hat{H}_n(\xi, \tau)$, may be defined as:

$$\left.\begin{aligned}
&\frac{\partial F}{\partial \tau}+\frac{\partial G}{\partial \xi}=-\frac{1}{\xi}G \\
&\frac{\partial G}{\partial \tau}+\frac{\partial}{\partial \xi}\left(\frac{G^2}{F}+F^2\right)=-\frac{1}{\xi}\frac{G^2}{F}-\gamma FG \\
&\frac{\partial H}{\partial \tau}+\frac{\partial}{\partial \xi}\left(\frac{GH}{F}\right)=-\frac{1}{\xi}\frac{GH}{F}
\end{aligned}\right. \quad \text{with} \quad \left.\begin{aligned}
F&=\hat{F}_n(\xi) \\
G&=\hat{G}_n(\xi) \\
H&=\hat{H}_n(\xi)
\end{aligned}\right\} \quad \text{at} \quad \tau=\tau_n \tag{11}$$

In order to approximate the fields F, G and H at the time $\tau=\tau_{n+1}$ in the non-homogeneous problem described in Eq. (11) an operator splitting technique, described in details by Martins-Costa and Saldanha da Gama (2001) is employed. It consists of a decomposition of the operator defined in Eq. (11) so that its merely hyperbolic part – namely the homogeneous associated system, is split away from its purely time evolutionary one – essentially an ordinary system. This technique gives rise to an initial approximation, obtained by advancing $\Delta\tau$ in time through the equations representing the homogeneous problem, by employing Glimm's method.

Once this approximation has been evaluated, the numerical approximation for the solution (F,G,H) at time $\tau=\tau_{n+1}$ is finally reached by advancing in time to solve the following time evolutionary problem, with the same step $\Delta\tau=\tau_{n+1}-\tau_n$ through equations:

$$\left\{\begin{aligned}
&\frac{\partial F}{\partial \tau}=-\frac{1}{\xi}G \\
&\frac{\partial G}{\partial \tau}=-\frac{1}{\xi}\frac{G^2}{F}-\gamma FG \\
&\frac{\partial H}{\partial \tau}=-\frac{1}{\xi}\frac{GH}{F}
\end{aligned}\right. \quad \text{with} \quad \left.\begin{aligned}
F&=\hat{F}_{n+1}(\xi) \\
G&=\hat{G}_{n+1}(\xi) \\
H&=\hat{H}_{n+1}(\xi)
\end{aligned}\right\} \quad \text{at} \quad \tau=\tau_n \tag{12}$$

by employing a first order Euler approximation, as follows:

$$\begin{aligned}
F=\hat{F}_{n+1}(\xi)&\approx\tilde{F}_{n+1}(\xi)-\left\{\frac{1}{\xi}\tilde{G}_{n+1}(\xi)\right\}\Delta\tau \\
G=\hat{G}_{n+1}(\xi)&\approx\tilde{G}_{n+1}(\xi)-\left\{\frac{1}{\xi}\frac{\left[\tilde{G}_{n+1}(\xi)\right]^2}{\tilde{F}_{n+1}(\xi)}-\gamma\tilde{F}_{n+1}(\xi)\tilde{G}_{n+1}(\xi)\right\}\Delta\tau \\
H=\hat{H}_{n+1}(\xi)&\approx\tilde{H}_{n+1}(\xi)-\left\{\frac{1}{\xi}\frac{\tilde{G}_{n+1}(\xi)\tilde{H}_{n+1}(\xi)}{\tilde{F}_{n+1}(\xi)}\right\}\Delta\tau
\end{aligned} \tag{13}$$

in which all quantities have been evaluated at τ_{n+1} and considering $\Delta\tau=\tau_{n+1}-\tau_n$.

The fields $\tilde{F}_{n+1}(\xi)$, $\tilde{G}_{n+1}(\xi)$ and $\tilde{H}_{n+1}(\xi)$ used as initial data in (13) are obtained by advancing $\Delta\tau$ in time via Glimm's method, through the following homogeneous problem:

$$\left\{\begin{aligned}
&\frac{\partial F}{\partial \tau}+\frac{\partial G}{\partial \xi}=0 \\
&\frac{\partial G}{\partial \tau}+\frac{\partial}{\partial \xi}\left(\frac{G^2}{F}+F^2\right)=0 \\
&\frac{\partial H}{\partial \tau}+\frac{\partial}{\partial \xi}\left(\frac{GH}{F}\right)=0
\end{aligned}\right. \quad \text{with} \quad \left.\begin{aligned}
F&=\hat{F}_n(\xi) \\
G&=\hat{G}_n(\xi) \\
H&=\hat{H}_n(\xi)
\end{aligned}\right\} \quad \text{at} \quad \tau=\tau_n \tag{14}$$

In other words, $\tilde{F}_{n+1}(\xi)$, $\tilde{G}_{n+1}(\xi)$ and $\tilde{H}_{n+1}(\xi)$ are the solutions of (14) evaluated at the time τ_{n+1}. The main idea behind Glimm's scheme (Smoller, 1983) is to appropriately gather the solution of as many Riemann problems as desired in order to successively march from time τ_n to τ_{n+1}. Glimm's scheme, specifically developed to deal with discontinuous problems, preserves the shock waves magnitude and position, within an uncertainty of $\Delta\xi$ (width of each step). Such

features are not found in the usual numerical procedures (e.g. finite elements and finite differences). Besides, Glimm's method presents a clear advantage of saving computer storage memory, when compared to other methodologies such as a finite element method associated with a shock capture procedure, however its limitation to one-dimensional problems is an important shortcoming. In order to employ this scheme a piecewise constant function is used to approximate the initial data, as follows:

$$\left.\begin{aligned} F &= \hat{F}_n(\xi) \approx F_{n_i} = \hat{F}_n(\xi_i + \theta_n \Delta\xi) \\ G &= \hat{G}_n(\xi) \approx G_{n_i} = \hat{G}_n(\xi_i + \theta_n \Delta\xi) \\ H &= \hat{H}_n(\xi) \approx H_{n_i} = \hat{H}_n(\xi_i + \theta_n \Delta\xi) \end{aligned}\right\} \qquad \text{at} \qquad \xi_i - \frac{\Delta\xi}{2} < \xi < \xi_i + \frac{\Delta\xi}{2} \tag{15}$$

in which θ_n is a number randomly chosen in the open interval $(-1/2\ ,\ +1/2)$ and $\Delta\xi$ is the width of each step ($\Delta\xi = \xi_{i+1} - \xi_i$).

The above approximations for the initial data give rise, for each two consecutive steps, to the following Riemann problem – whose detailed solution is presented in Martins-Costa and Saldanha da Gama (2003):

$$\begin{aligned} &\frac{\partial F}{\partial \tau} + \frac{\partial G}{\partial \xi} = 0 \\ &\frac{\partial G}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\frac{G^2}{F} + F^2\right) = 0 \\ &\frac{\partial H}{\partial \tau} + \frac{\partial}{\partial \xi}\left(\frac{GH}{F}\right) = 0 \end{aligned} \qquad \begin{aligned} &(F,G,H) = (F_{n_i}, G_{n_i}, H_{n_i}) \qquad \text{for } t = t_n\ ,\ -\infty < \xi < \xi_i + \frac{\Delta\xi}{2} \\ &(F,G,H) = (F_{n_{i+1}}, G_{n_{i+1}}, H_{n_{i+1}}) \qquad \text{for } t = t_n\ ,\ \xi_{i+1} - \frac{\Delta\xi}{2} < \xi < \infty \end{aligned} \tag{16}$$

Denoting by $F^R_{n_i}$ $G^R_{n_i}$, and $H^R_{n_i}$ the generalized solution of the Riemann problem (16), the approximation for the solution of (14) at the time τ_{n+1} is given as follows:

$$\begin{aligned} F &= \hat{F}_{n+1}(\xi) \approx F^R_{n_i}(\xi, \tau_{n+1}) \qquad \text{for} \quad \xi_i < \xi < \xi_{i+1} \\ G &= \hat{G}_{n+1}(\xi) \approx G^R_{n_i}(\xi, \tau_{n+1}) \qquad \text{for} \quad \xi_i < \xi < \xi_{i+1} \\ H &= \hat{H}_{n+1}(\xi) \approx H^R_{n_i}(\xi, \tau_{n+1}) \qquad \text{for} \quad \xi_i < \xi < \xi_{i+1} \end{aligned} \tag{17}$$

In order to prevent interactions among nearby shocks of adjacent Riemann problems, the time step $\Delta\tau$, and consequently, τ_{n+1} must be chosen in such a way that the Courant-Friedrich-Lewy (Smoller, 1977) condition is satisfied:

$$\tau_{n+1} - \tau_n \le \frac{\Delta\xi}{2|\lambda|_{\max}} \tag{18}$$

where $|\lambda|_{\max}$ is the maximum (in absolute value) propagation speed, considering all the Riemann problems.

4. Numerical results

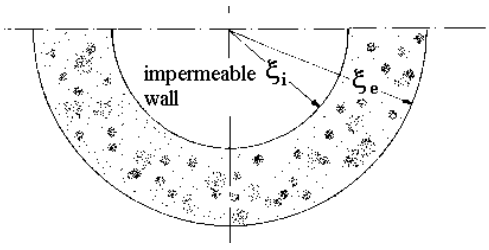

Figure 1. Problem statement.

This section presents some results obtained by employing the preliminary model described in the previous sections to study the radial propagation of a pollutant in soils, starting from a cylindrical contaminated well. These results have been obtained by approximating the problem stated in Eqs. (9) with initial data given by a step function, subjected to a boundary condition characterizing an impermeable wall at the inner surface of the cylindrical shell porous matrix, depicted in Fig. (1), which is given by

$$\xi = 0 \;\Rightarrow\; u = 0 \tag{19}$$

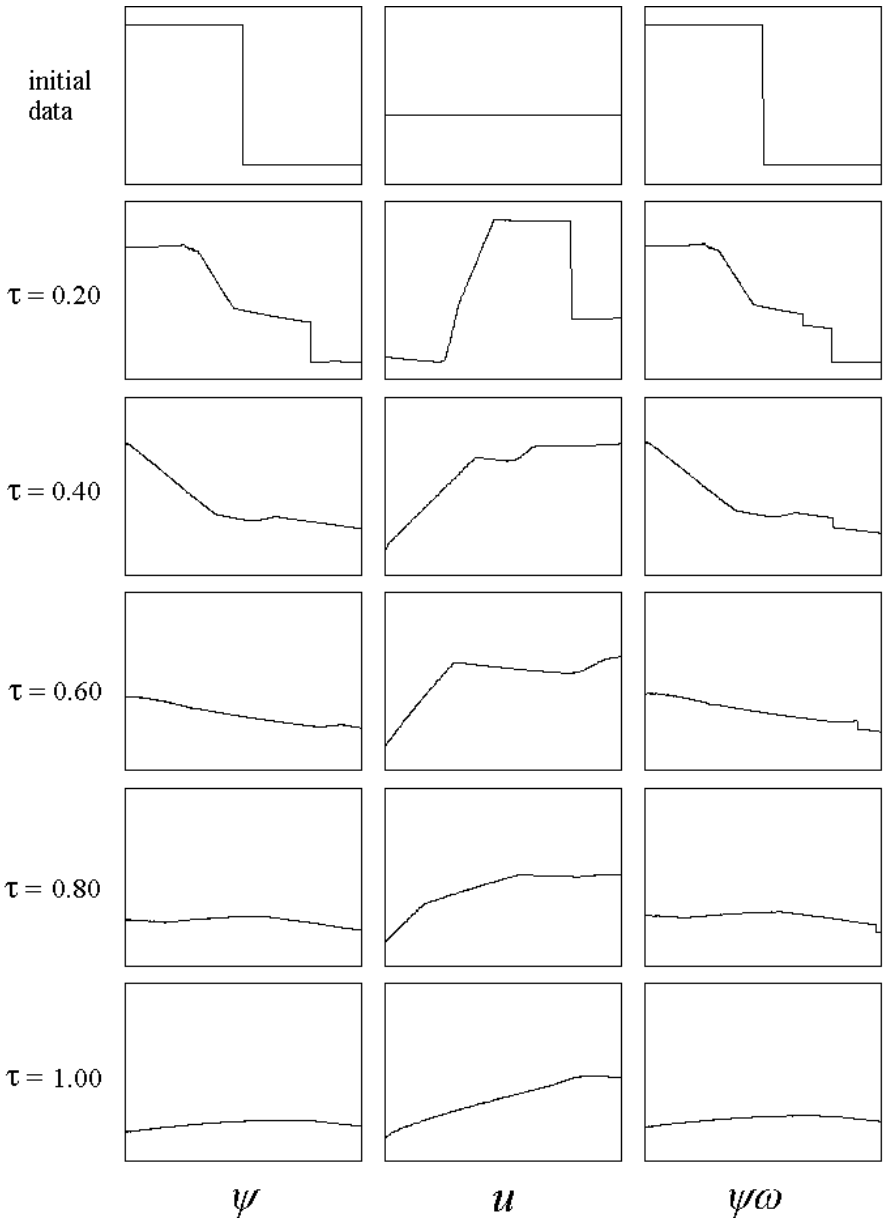

Figure 2. Saturation, binary mixture velocity and pollutant concentration variation with radial position for $\gamma = 1$. Each line: distinct τ – from left to right, vertical axis: numerical values of ψ, u and $\psi\omega$; horizontal axis: ξ. Initial data: step function for ψ and ω and constant u.

Some selected results are presented in Figs. (2) to (4), each of them consisting of a set of six lines and three columns of graphs, each line representing a distinct time instant – the first one being the initial condition – and each column representing the behavior of a distinct variable. The evolution of saturation ψ, fluid constituent (binary mixture) velocity u and pollutant concentration $\psi\omega$ along with radial position for five selected time instants, starting from the initial data depicted in the first line, is shown. All these results have been obtained by employing Glimm's

difference scheme combined with an operator splitting technique to account for the non-homogeneous portion of the hyperbolic operator with 600 steps for each time advance. In all computations the temporal and spatial increments considered were such that $\Delta t / \Delta x = 10^{-4}$, satisfying the Courant-Friedrich-Lewy condition expressed by Eq. (18).

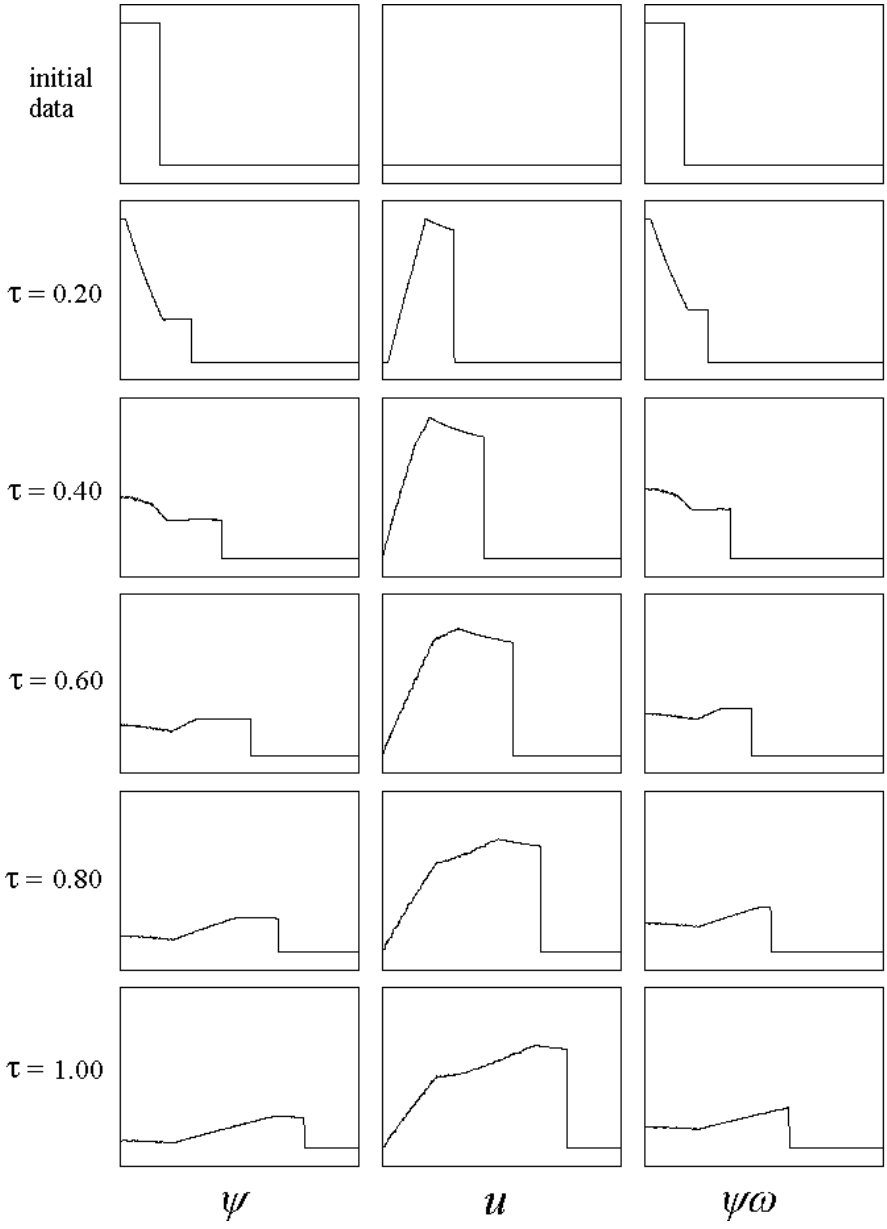

Figure 3. Saturation, binary mixture velocity and pollutant concentration variation with radial position for $\gamma = 1$. Each line: distinct τ – from left to right, vertical axis: numerical values of ψ, u and $\psi\omega$; horizontal axis: ξ. Initial data: step function for ψ and ω and zero u.

The qualitative behavior of saturation, fluid constituent velocity and pollutant concentration, expressing the expected features of the hyperbolic operator employed to describe the problem, is displayed from left to right, the vertical axis corresponding to their numerical value and the horizontal axis to the spatial coordinate ξ. In all the depicted graphs the left-hand side corresponds to ξ_i, the cylindrical shell porous matrix internal radius ($\xi = 0$) while at the right-hand side ξ_e represents the external radius ($\xi = 1$). A convenient normalization, accounting for maximum and minimum values of ψ, u and ω was performed in such a way that the minimum and maximum displayed values correspond to zero (or a minimum) and unit values for ψ, u and ω.

Figure 2 shows the pollutant propagation in the radial direction from the internal radius ξ_i to the external ξ_e, the pollutant being initially concentrated in the internal half of the cylindrical shell. The binary mixture velocity, initially constant, presents an increment caused by the jump in the saturation. The decrease in the velocity is caused by two distinct factors: the presence of the Darcian term, as well as a geometrical factor, since the flow area increases linearly with the radius of the cylindrical shell. Also, it may be observed that all the considered region between ξ_i and ξ_e suffers the influence of the perturbation caused by the saturation and pollutant concentration.

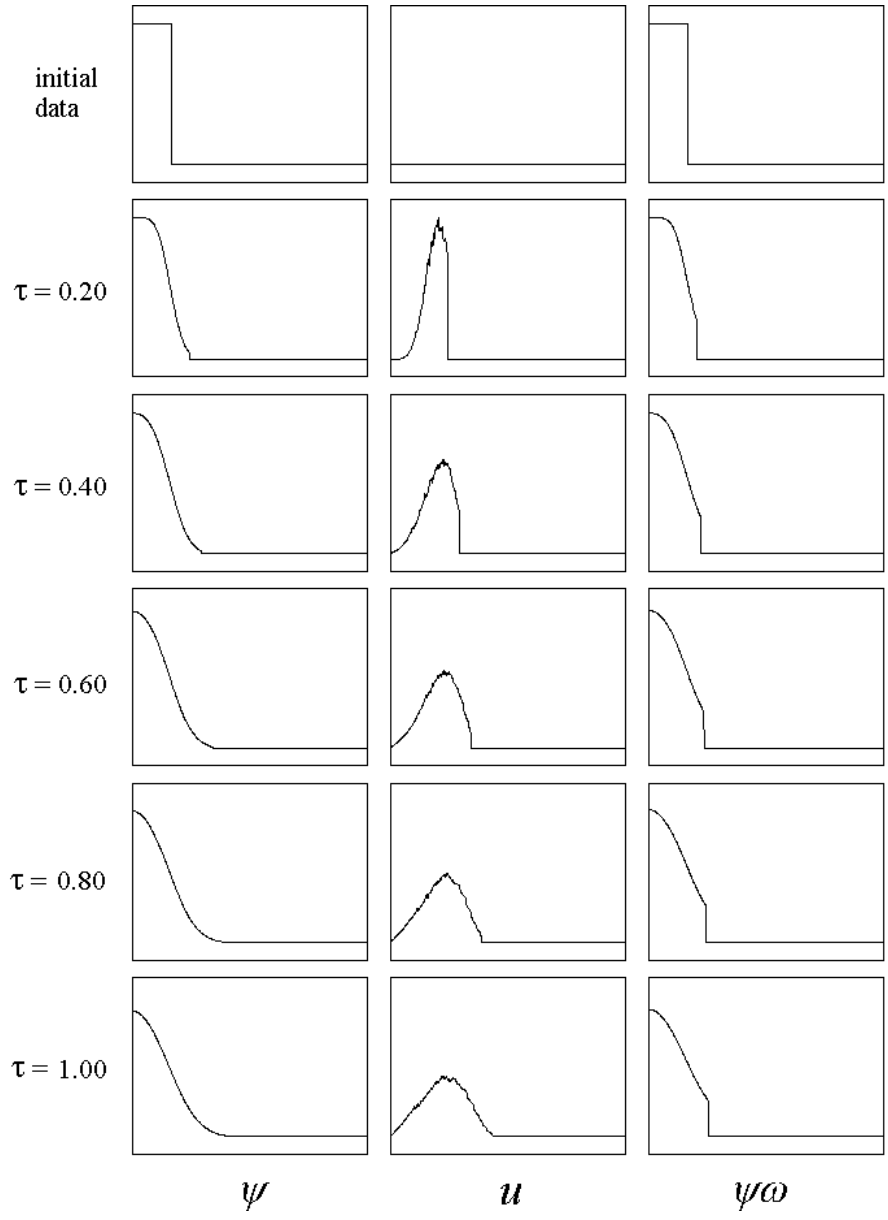

Figure 4. Saturation, binary mixture velocity and pollutant concentration variation with radial position for $\gamma = 100$.
Each line: distinct τ – from left to right, vertical axis: numerical values of ψ, u and $\psi\omega$; horizontal axis: ξ.
Initial data: step function for ψ and ω and zero u.

Figures (3) and (4) show the evolution of the pollutant propagation in the radial direction, starting from the same initial data – namely zero velocity and jumps for the saturation ψ and pollutant concentration ω, with the pollutant being initially concentrated in a neighborhood of ξ_i. Since the only varying parameter is the Darcian term coefficient, with $\gamma = 1$ in Fig. (3) – the same value used in Fig. (2) – and $\gamma = 100$ in Fig. (4), a comparison between these two

figures shows a strong influence of the Darcian term coefficient not only smoothening the shocks but also decreasing the propagation speed of perturbation. Actually, in Fig. (4) only $\psi\omega$ shows a shock propagation front.

An important feature, present in all depicted results, is that the discontinuities for the variables ψ and u are in the same spatial position. The numerical method accuracy (see Martins-Costa and Saldanha da Gama (2004) and references therein) is mathematically ensured, all quantities being globally preserved in such a way that the maximum error concerning the shock position is of the order of magnitude of the step width while the shock amplitude is preserved – no shock dissipation being present.

5. Final remarks

A preliminary local model for the flow of a Newtonian fluid containing a pollutant through a wellbore was built under a mixture theory approach and, after considering the flow on radial direction only, the resulting set of three nonlinear partial differential equations has been simulated by combining Glimm's scheme with an operator splitting technique to account for the non-homogeneous part of the hyperbolic operator.

Glimm's method, besides preserving shock waves magnitude and position, is a convenient tool for solving one-dimensional nonlinear problems, exhibiting features such as low storage costs and low computational effort, when compared to other numerical procedures to approximate nonlinear problems. Besides, combined with an operator splitting technique, this numerical methodology allows the accurate approximation of a nonlinear system of non-homogeneous partial differential equations representing mathematically the flow of a polluted fluid through a wellbore.

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