Proceedings of the 10o Brazilian Congress of Thermal Sciences and Engineering -- ENCIT 2004
Braz. Soc. of Mechanical Sciences and Engineering -- ABCM, Rio de Janeiro, Brazil, Nov. 29 -- Dec. 03, 2004

THERMAL BUCKLING AND POST-BUCKLING OF SLENDER RODS WITH ENDS SUBJECTED TO DIFFERENT BOUNDARY CONDITIONS

Rafael Familiar Solano
PETROBRAS, Engineering Department
rsolano@petrobras.com.br

Murilo Augusto Vaz
COPPE/UFRJ, Ocean Engineering Program
murilo@peno.coppe.ufrj.br

Abstract. *This paper presents mathematical formulation, critical buckling temperature and analytical and numerical solutions for the thermal post-buckling behavior of slender rods subjected to uniform thermal load. The material is assumed to be linearly elastic, homogeneous and isotropic. Furthermore, large displacements are considered hence the formulation is geometrically non-linear. Three different boundary conditions are assumed: (i) double-hinged non-movable, (ii) hinged non-movable at one end, whereas at the other end longitudinal displacement is constrained by a linear spring, and (iii) double-fixed non-movable. The governing equations are derived from geometrical compatibility, equilibrium of forces and moments, constitutive equations and strain-displacement relation, yielding a set of six first-order non-linear ordinary differential equations with boundary conditions specified at both ends, which constitutes a complex boundary value problem. The buckling and post-buckling solutions are respectively accomplished assuming infinitesimal and finite rotations. The results are presented in non-dimensional graphs for a range of temperature gradients and different values of slenderness ratios. It is shown that this parameter governs the buckling and post-buckling behavior. The influence of the boundary conditions is evaluated through graphic results for deformed configuration, maximum deflection, maximum inclination angle and maximum curvature in the rod.*

Keywords: *Elastic Rods, Thermal Buckling, Thermal Post-Buckling.*

1. Introduction

There are many practical cases where buckling and post-buckling of slender rods may occur. Such a very narrow relationship between the thermal buckling of slender components - such as railroad tracks, concrete road pavements, optical fibers, satellite tethers or subsea and buried pipelines - and the buckling of rods has long been recognized. It is therefore of practical design interest to employ simplified analysis. Pipeline instability analysis has been studied greatly, and firstly it has been made reference to similar problems occurred with raiload track (Martinet, 1936 and Kerr, 1974). Analytical and numerical modelling of the buckling response of offshore pipelines has progressed rapidly over the last few years, broadly from the classical analysis (Hobbs, 1984 and Hobbs and Liang, 1989) that has been extensively accepted for industrial design. Similar studies were presented by Ju and Kyriakides (1998), Chiou and S. –Y. Chi (1996) and Taylor and Gan (1996). The recent increase of the necessity of high temperature flowlines and the lack of publications about the subject unleashed the interest on the study of this phenomenon. Several papers that describe the structural behavior of pipelines subjected to the action of thermal loading are important to this study.

The problem of elastic stability of rods subjected to mechanical and thermal compressive loads has been well studied since Bernoulli, Euler and Lagrange investigated the classical problem of the *elastica*, i.e., the equilibrium configurations of inextensible rods under axial compression. Love's (1944) seminal textbook on theory of mathematical elasticity has been extensively used in many fields of applied mechanics, establishing the basis for most research on the equilibrium of elastic rods. Some papers were published on buckling and post-buckling behavior obtaining solutions for the differential equation that governs the elastic line of an initially straight slender rod (the *elastica* problem) subjected to different compressive loads and boundary conditions (Theocaris and Panayotounakos, 1982; Stemple, 1990; Wang, 1997; Filipich and Rosales, 2000 and Vaz and Silva, 2002).

The problem of elastic stability of rods subjected to thermal loads and mechanical compressive loads are substantially different and in fact not as many articles have been published regarding thermal buckling of rods. Buckling and post-buckling behavior in the sense of Koiter were treated within the framework of the general branching theory of discrete systems. Coffin and Bloom (1999) developed an elliptic integral solution for the post-buckling response of a linear-elastic and hygrothermal beam fully restrained against axial expansion. They assumed linear thermal strain-temperature relationship and solved the set of differential equations for the undeformed configuration, hence two coupled integral elliptic equations needed to be simultaneously solved. Based on the exact non-linear geometric theory for extensible rods and using a shooting method, a computational analysis for the thermal post-buckling behavior of rods with axially non-movable pinned-pinned ends as well as fixed-fixed ends was proposed by Li and Cheng (2000). More recently, Li et al. (2002) presented a mathematical model for the post-buckling of an elastic rod with pinned-fixed ends when a quasi-static increasing temperature is applied. Using the shooting method in conjunction with the concept of analytical continuation, the non-linear boundary value problem consisting of ordinary differential equations was numerically solved. The results showed that the critical buckling temperature and the post-buckled rod configuration were sensitively influenced by the slenderness ratio. Finally, Vaz and Solano (2003 and 2004) developed a closed-form analytical solution via uncoupled elliptical integrals for the buckling and post-buckling analysis of slender elastic rods subjected to uniform thermal loads.

This paper investigates the buckling and post-buckling response of an initially straight slender rod made of linear elastic material. A temperature gradient is assumed uniform along the rod and expansion is prevented by different boundary conditions. The analytical solution is obtained by uncoupled elliptic integrals, which are derived from the governing equations in the deformed configuration, and numerical technique employs a classical Runge-Kutta high order solution to solve the set of non-linear ordinary differential equations. This study may be qualitatively expanded to pipelines and other slender structures subjected to thermal loads.

2. Mathematical Formulation

Consider a uniformly heated slender rod with ends subject to different boundary conditions in its initial and buckled configurations, as shown in Fig. 1. Whereas in Fig. 1a the rod has non-movable double-hinged ends (bi-pinned), in Fig. 1b one edge is considered non-movable and the other edge is limited by a linear spring of stiffness constant K_S, which restrains the longitudinal expansion. Finally in Fig. 1c the rod has non-movable double-fixed (bi-clamped) ends.

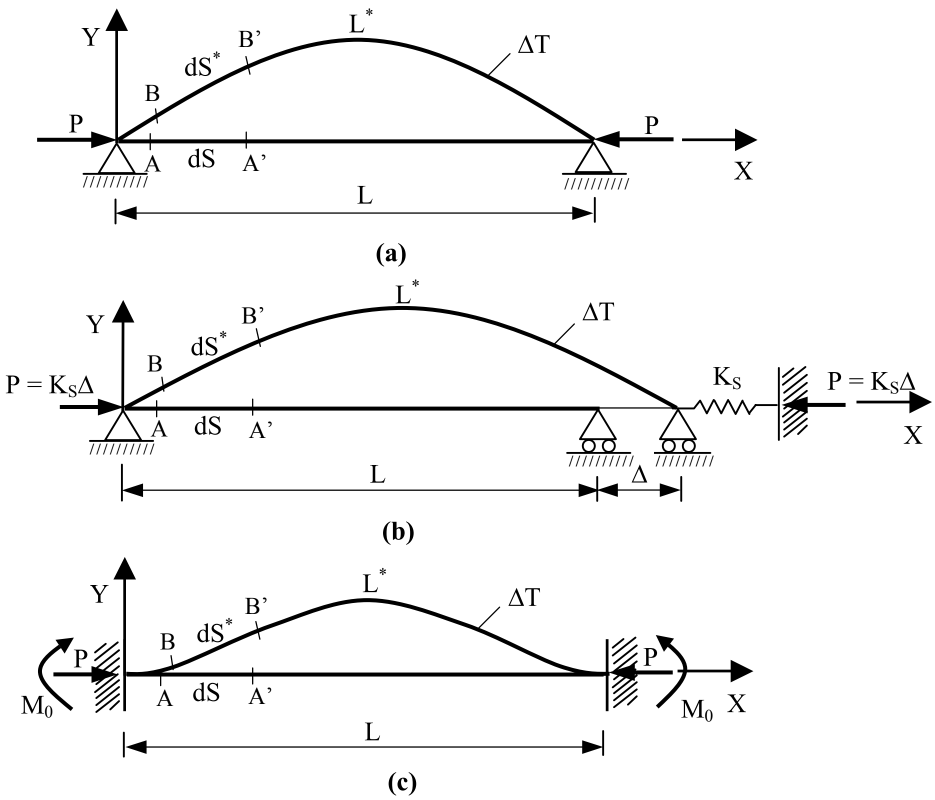

Figure 1. Schematic of a Heated Elastic Rod.

In this paper (*X,Y*) constitute the rod central line Cartesian coordinates, ΔT is the uniform temperature gradient, P is the compressive load arising from the expansion constraint, M_0 is the end constrained moment, S is the arc-length, S^* is the deformed arc-length, L and L^* are respectively the initial and deformed rod length, and Δ is the edge lateral displacement.

The governing equations are derived from the geometrical compatibility, equilibrium of forces and moments, constitutive equations and strain-displacement relation, following the development presented by Vaz and Solano (2003 and 2004) and considering the infinitesimal deformed element of the rod (see Fig. 2).

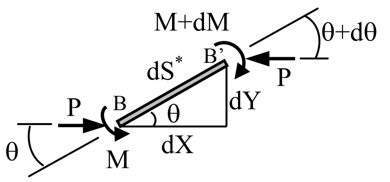

Figure 2. Infinitesimal Element of the Deflected Rod.

Where M is the bending moment, θ is the angle formed by the curve tangent and the longitudinal axis.

Furthermore, non-dimensional variables may be conveniently defined by using the following relations: $L^* = l^* L$ $S = s\,L$, $X = x\,L$, $Y = y\,L$, $S^* = s^* L$, $\mathrm{K} = \kappa / L$, $\lambda^2 = L^2 A / I$, $P = pEI / L^2$, $M_0 = mEI / L$, $K_S = k_s\, EI / L^3$, $\Delta = \delta\, L$ and $\Delta T = \Delta t / \lambda^2 \alpha$. Where: K is the curvature, E is the Young's modulus, I is the cross-sectional second moment of inertia, A is the cross-sectional area, λ is the rod slenderness ratio and α is the thermal expansion coefficient.

Finally, assuming linear elastic, homogeneous and isotropic materials (constitutive relations given by Hooke`s Law), and still considering the state of pure bending, the governing equations for slender rods subjected to a uniform temperature gradient are written as:

$$\frac{dx}{ds^*} = \cos\theta \tag{1.1}$$

$$\frac{dy}{ds^*} = \sin\theta \tag{1.2}$$

$$\frac{d\theta}{ds^*} = \kappa \tag{1.3}$$

$$\frac{ds}{ds^*} = \frac{1}{(1+\varepsilon)} \tag{1.4}$$

$$\frac{d\kappa}{ds^*} = -p\sin\theta \tag{1.5}$$

$$\frac{dp}{ds^*} = 0 \tag{1.6}$$

Where:

$$\varepsilon = \frac{\Delta t}{\lambda^2} - \frac{p}{\lambda^2}\cos\theta \tag{1.7}$$

is the central line strain and the constant, defined as the ratio between the elongation in deformed configuration and its initial length.

In summary, a slender rod tends to expand when it is subjected to a temperature gradient Δt and, consequently, a compressive load p appears if movement of the ends is constrained. Hence, the total strain is given by the addition of the thermal strain and the strain due to the compressive load ($\varepsilon = \varepsilon_T + \varepsilon_c$). The first term on the right hand side of Eq. (1g) defines the thermal strain for materials whose strain-temperature dependence is linear.

Associated to the governing equations (1) the following boundary equations should be fulfilled:

$$x(0) = y(0) = \kappa(0) = x(l^*) - 1 = y(l^*) = \kappa(l^*) = 0 \tag{2.1}$$

$$x(0) = y(0) = \kappa(0) = x(l^*) - 1 - \delta = y(l^*) = \kappa(l^*) = 0 \tag{2.2}$$

$$x(0) = y(0) = \theta(0) = x(l^*) - 1 = y(l^*) = \theta(l^*) = 0 \tag{2.3}$$

Where Eqs. (2.1), (2.2) and (2.3) represent respectively the bi-pinned, pined-axially constrained by a linear spring and boundary bi-clamped boundary conditions.

Hence the influence of the slenderness ratio and degree of edge mobility (i.e., spring stiffness) on the critical buckling load and temperature and on the rod post-buckled deformed configuration may be analytical and numerically calculated.

3. Critical Buckling Temperature

The determination of the critical buckling load is found applying the equilibrium equations to the rod element in an infinitesimal slightly deformed configuration. As the rotation θ is assumed small compared to unity, $\cos\theta \cong 1$ and $\sin\theta \cong \theta$. Consequently, the governing equation may be reduced to:

$$\frac{d^4 y}{dx^4} + p\frac{d^2 y}{dx^2} = 0 \tag{3}$$

General solution for the homogeneous differential equation (3) with constant coefficients is quickly found:

$$y(x) = C_1 \sin(\sqrt{p}\, x) + C_2 \cos(\sqrt{p}\, x) + C_3 x + C_4 \qquad (4)$$

3.1. Double-hinged non-movable (bi-pinned) ends

Application of boundary conditions for rods with double-hinged non-movable ends yields $C_2 = C_3 = C_4 = C_1 \sin(\sqrt{p}) = 0$, and to avoid trivial solution C_1 must be different from zero, which can be satisfied if $\sin(\sqrt{p}) = 0$ and $\sqrt{p} = n\pi$, where n is a positive integer. The smallest eigenvalue in this case corresponds to $n = 1$, i.e., the first buckling mode corresponds to:

$$p_c = \pi^2 \qquad (5)$$

Subjected to a uniform temperature increase, the rod tends to expand, but until it reaches the critical buckling load, its strain is zero $(\varepsilon = 0)$, hence:

$$\Delta t - p = 0 \qquad (6)$$

Equation (5) can be substituted in Eq. (6) to find the critical buckling temperature:

$$\Delta t_c = p = \pi^2 \qquad (7)$$

3.2. At one end longitudinal displacement is constrained by a linear spring

It is known that $p = k_s \delta$, so applying the boundary conditions for the rod: hinged non-movable at one end, and at the other end longitudinal displacement constrained by a linear spring, yields $C_2 = C_3 = C_4 = C_1 k_s \delta \sin[(1+\delta)\sqrt{k_s \delta}] = 0$. To avoid a trivial solution $(1+\delta)\sqrt{k_s \delta} = n\pi$ where n is a positive integer. The smallest eigenvalue corresponds to $n = 1$, hence the critical lateral displacement for first buckling mode is given by:

$$\delta_c = \frac{(w-2)^2}{6w} \qquad (8)$$

where: $w = \left[8 + 108\frac{\pi^2}{k_s} + \frac{12\pi}{k_s}\sqrt{12k_s + 81\pi^2} \right]^{1/3}$

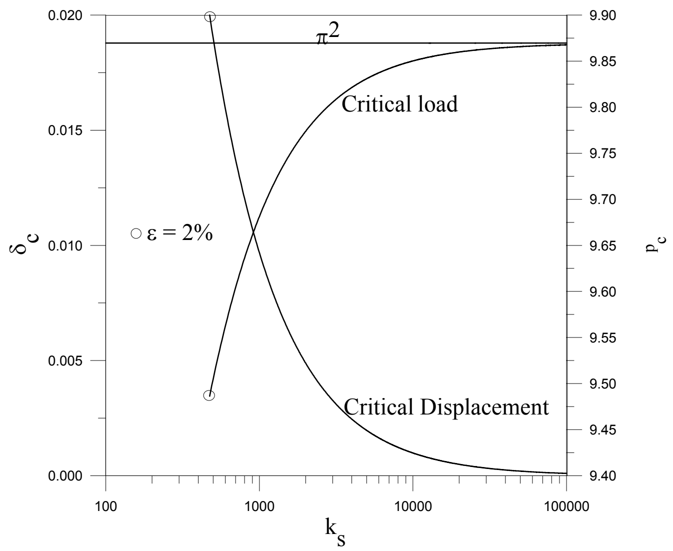

Figure 3. Critical Displacement and Load as a Function of the Spring Stiffness.

It is interesting to show the influence of the spring stiffness on the critical buckling load. In Fig. 3 the critical displacement and load are presented as a function of the spring stiffness. As the spring stiffness increases the end displacement and the buckling load tend to zero and π^2, respectively. On the other hand as k_s tends to zero, δ_c grows to infinity and the critical buckling load tends to zero. However, note that the rod strain has been limited to 2%, which gives $k_s = 474.32$, $\delta_c = 0.02$ and $p_c = 9.4864$.

Until the critical buckling load (p_c) is reached a uniformly heated rod tends to expand maintaining its straight configuration (i.e., $\varepsilon = \delta_c$), hence from Eq. (1.7):

$$\Delta t = \delta_c (k_s + \lambda^2) \tag{9}$$

Note that three parameters control the critical buckling temperature: the spring stiffness k_s and the rod slenderness ratio λ, the critical lateral displacement δ_c being a function of k_s through Eq. (8).

The physical and geometrical rod properties should be carefully selected to ensure practical and real meaning to the analysis, and concomitantly avoiding violation of assumptions included in the mathematical formulation. Therefore, high temperatures and strains should not be considered. Furthermore, the parametric study was conducted for rod slenderness ratios $\lambda = 50, 100, 150$ and 200.

Figure 4 presents the critical buckling temperature as a function of the spring stiffness for four rod slenderness ratios. As the spring stiffness increases the boundary conditions tend to double-hinged non-movable and as expected the critical temperature tends to π^2.

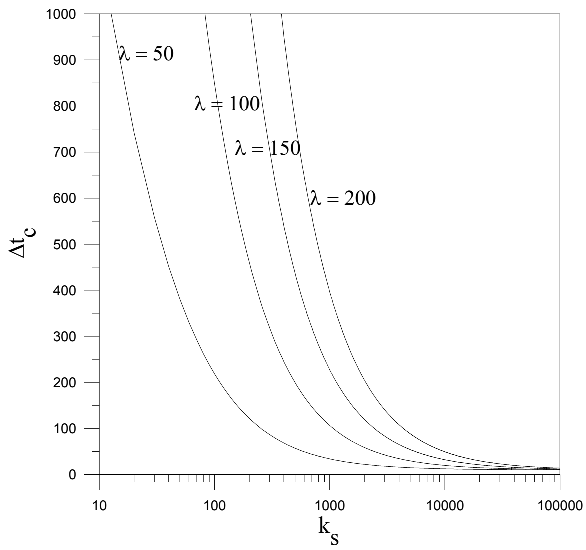

Figure 4. Critical Temperature as a Function of the Spring Stiffness.

3.3. Double-fixed non-movable (bi-clamped) ends

Application of boundary conditions for rods with double-fixed non-movable ends yields a homogeneous equation system. For a nontrivial solution of this homogeneous equation system to exist, the determinate of the coefficients of the C_1, C_2, C_3 and C_4 must equal zero. After some rearrangement, the determinate is found to be the product of two factors with the result that either $\sin(\sqrt{p}/2) = 0$ or $\tan(\sqrt{p}/2) = \sqrt{p}/2$. The smallest nonzero value of $\sqrt{p}/2$ for which $\sin(\sqrt{p}/2) = 0$ is π, whereas the smallest value for which $\tan(\sqrt{p}/2) = \sqrt{p}/2$ is 4.49. Consequently, the smallest eigenvalue is $\sqrt{p} = 2n\pi$, where n is a positive integer. The smallest eigenvalue in this case corresponds to $n = 1$, i.e., the first buckling mode corresponds to:

$$p_c = 4\pi^2 \tag{10}$$

Subjected to a uniform temperature increase, the rod tends to expand, but until it reaches the critical buckling load, its strain is zero $(\varepsilon = 0)$, hence:

$$\Delta t - p = 0 \tag{11}$$

Equation (10) can be substituted in Eq. (11) to find the critical buckling temperature:

$$\Delta t_c = p = 4\pi^2 \tag{12}$$

Equation (12) indicates that two parameters control the critical buckling temperature, the rod slenderness ratio λ.

4. Analytical and Numerical Solutions

4.1. Analytical Solution

For the boundary conditions: (i) double-hinged non-movable and (ii) hinged non-movable at one end, whereas at the other end longitudinal displacement is constrained by a linear spring ($p = k_s \delta$), a closed-form analytical solution for the thermal post-buckling of slender elastic rod uniformly heated is developed next via complete elliptical integrals derived from the governing equations in the deformed configuration, following similar works developed by Vaz and Solano (2003 and 2004). The material is assumed linearly elastic and its thermal strain-temperature relationship is linear. It is more convenient to work with the slope angle θ (Bažant and Cedolin (1991)), so the non-dimensional differential Eqs. (1.3) and (1.5) yield:

$$\frac{d^2\theta}{ds^{*2}} = -p\sin\theta \tag{13}$$

The solution of the non-linear ordinary differential equation (13) had been earlier solved by Lagrange (a kinetic analogy of columns; Love [8]). Integrating Eq. (13) and applying the boundary conditions at the ends of the rod (i.e., $\theta(0) = -\theta(l^*) = \beta$ and $\kappa(0) = \kappa(l^*) = 0$) yield:

$$\frac{d\theta}{ds^*} = -\sqrt{2p(\cos\theta - \cos\beta)} \tag{14}$$

Recurring to familiar trigonometric identities to rewrite the Eq. (14), separating and changing variables ($\sin\theta/2 = c\sin\phi$, where $c = \sin\beta/2$) and after some algebraic manipulation followed by an integration yield the deformed rod length:

$$l^* = \frac{2}{\sqrt{p}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-c^2\sin^2\phi}} \tag{15}$$

The slender rod deflected configuration may be obtained from the non-dimensional Eqs. (1.1) and (1.2) calculating the x and y coordinates respectively:

$$x = \frac{1}{\sqrt{p}}\int_{\phi_0}^{\pi/2}\frac{1-2c^2\sin^2\phi}{\sqrt{1-c^2\sin^2\phi}}d\phi \tag{16.1}$$

$$y = \frac{2c}{\sqrt{p}}\cos\phi_0 \tag{16.2}$$

And $-\pi/2 \le \phi_0 \le \pi/2$. Since $\kappa = -p\,y$ the rod curvature κ at the deformed configuration may now be readily obtained:

$$\kappa = -2c\sqrt{p}\cos\phi_0 \tag{17}$$

For the boundary condition double-hinged non-movable, symmetry implies that the point of maximum displacement occurs for $x(l^*/2) = 1/2$, so one may calculate p for this condition as a simple application of Eq. (16.1):

$$p = \left[2\int_0^{\pi/2}\frac{1-2c^2\sin^2\phi}{\sqrt{1-c^2\sin^2\phi}}d\phi\right]^2 \tag{18.1}$$

Similarly, for one edge limited by a linear spring, symmetry implies that the point of maximum displacement occurs for $x(l^*/2) = (1+\delta)/2$, so one may calculate δ for this condition as a simple application of Eq. (16.1):

$$\delta = \frac{(W-2)^2}{6W} \qquad (18.2)$$

where: $W = \left[8 + 108\frac{I}{k_s} + \frac{12\sqrt{I}}{k_s}\sqrt{12k_s + 81I} \right]^{1/3}$ and $I = \left(2\int_0^{\pi/2} \frac{1-2c^2\sin^2\phi}{\sqrt{1-c^2\sin^2\phi}} d\phi \right)^2$

Therefore for each deformed configuration (which is related to a temperature gradient), i.e., for a given slope β, $c = \sin\beta$ is calculated, and consequently p ($p = k_s\delta$ for one edge limited by a linear spring) from Eqs. (18.1) and (18.2). Finally, it is possible to find the coordinates (x, y) and curvature κ along the rod from Eqs. (16.1), (16.2) and (17).

The temperature gradient associated with the deformed configuration may be obtained considering the Eq. (1.7). Thus:

$$\Delta t = \lambda^2(l^* - 1) + 2\sqrt{p}\int_0^{\pi/2} \frac{(1-2c^2\sin^2\phi)^2}{(1-c^2\sin^2\phi)^{1/2}} d\phi \qquad (19)$$

This expression may be readily evaluated once p and L^* are known.

4.2. Numerical Solution

It is difficult to obtain analytical solutions to the boundary value problem, as slender rods with bi-clamped ends. The numerical solution for slender rods with (iii) double-fixed non-movable ends (bi-clamped), i.e., the solution of Eqs. (1.1)-(1.7) requires specification of six boundary conditions, given by Eqs. (2.3), as p is not known a priori. Furthermore, three boundary conditions are given at one end while more three are specified at the other end, which characterizes a two-point boundary value problem. Several techniques have been employed for this problem (e.g. finite element methods, finite difference schemes and energy methods). Solutions via the shooting method with direct integration are conveniently employed in liner or non-linear problems when only one parameter is required for interpolation but they become rather complex if two conditions are sought in non-linear systems.

However, a simple but robust way to transform a boundary into an initial value problem is available in Mathcad (2000) through the following procedure: (a) initial missing values are guessed; (b) the boundary value endpoints are specified; (c) the set of differential equations are defined; (d) a load function which returns the initial conditions is established; (e) a score function to measure the distance between terminal conditions and desired terminal conditions is employed; (f) the equivalent initial conditions are finally calculated. From this point, a classical Runge-Kutta high order solution may be employed to solve the set of non-linear ordinary differential equations.

5. Post-Buckling Results Analysis

The closed-form solution is implemented through a computational program developed in the mathematical software Mathcad (2000) and a parametric study is carried out with the purpose of analysing the results. The most significant results regarding the phenomenon of rod thermal post-buckling with (i) bi-pinned (double-hinged non-movable) and (iii) bi-clamped (double-fixed non-movable) ends are presented for typical values of slenderness ratio: compressive load (Fig. 5a), maximum deflection (Fig. 5b), maximum inclination angle (Fig. 5c) and maximum curvature (Fig. 5d).

Once the critical buckling load is reached and the temperature is progressively increased, the compressive force considerably decreases, as it can be observed at Fig. 5. The maximum rod deflection ($y_{\max}$), which occurs at $x(l^*/2) = 1/2$, increases with temperature as shown at Fig. 5b. The maximum inclination angle ($\theta_{\max}$) also increases with temperature (see Fig. 5c) but it occurs at the rod ends. The maximum curvature (Fig. 5d) occurs at the middle of the rod and also increases, in modulus, as temperature is progressively increased.

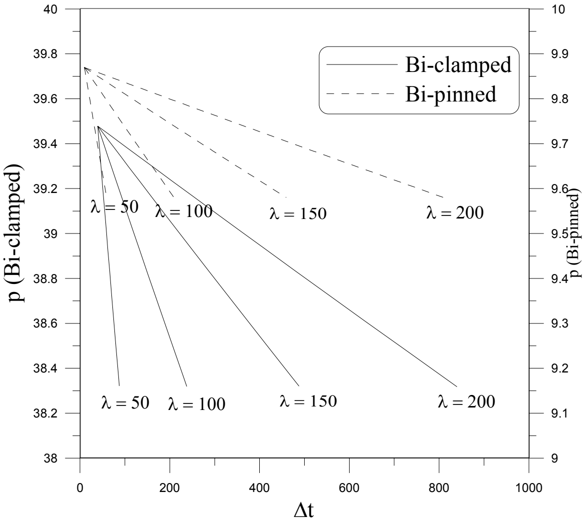

Figure 5a. Compressive Load.

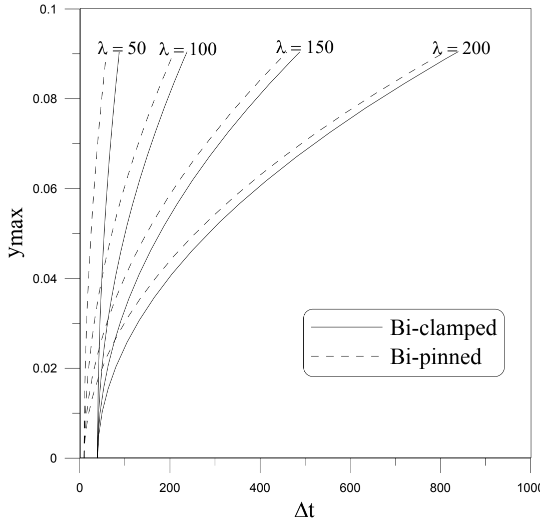

Figure 5b. Maximum Deflection.

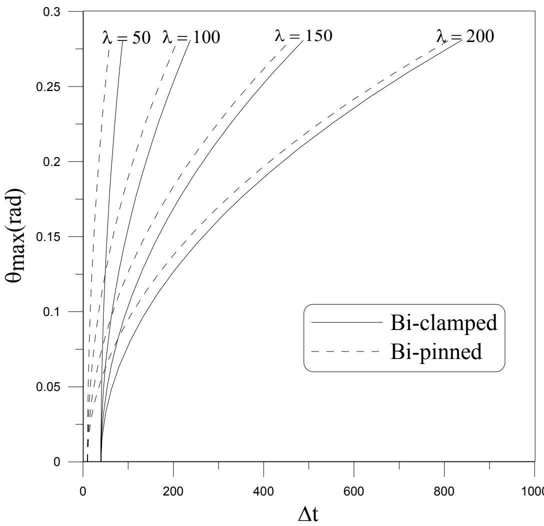

Figure 5c. Maximum Angle.

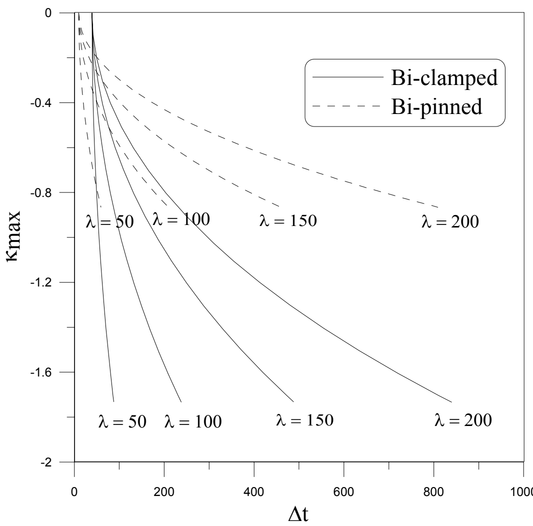

Figure 5d. Maximum Curvature.

Similarly, the results regarding the rod thermal post-buckling with (ii) one end non-movable whereas the other end is limited by a linear spring are presented for typical values of slenderness ratio and spring stiffness in the Figs. 6a to 6d. It is important to remind that the maximum rod deflection (y_{max}) occurs at $x(l^*/2) = (1+\delta)/2$.

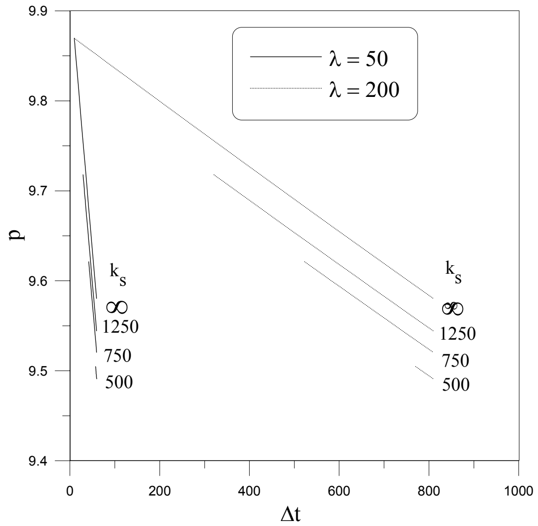

Figure 6a. Compressive Load.

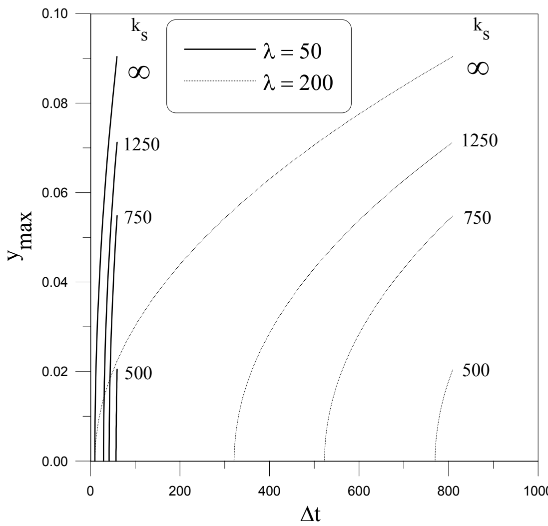

Figure 6b. Maximum Deflection.

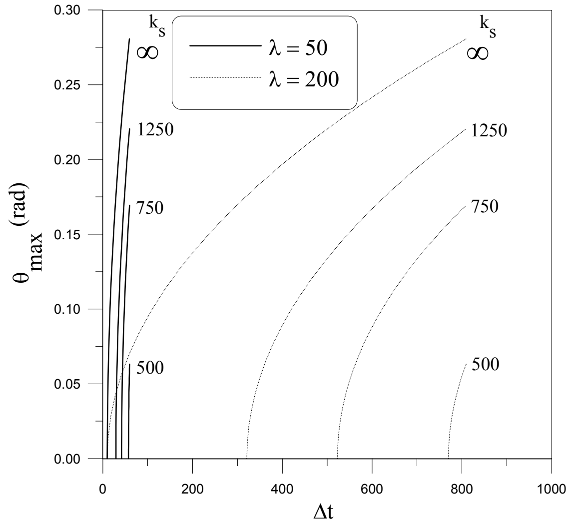

Figure 6c. Maximum Angle.

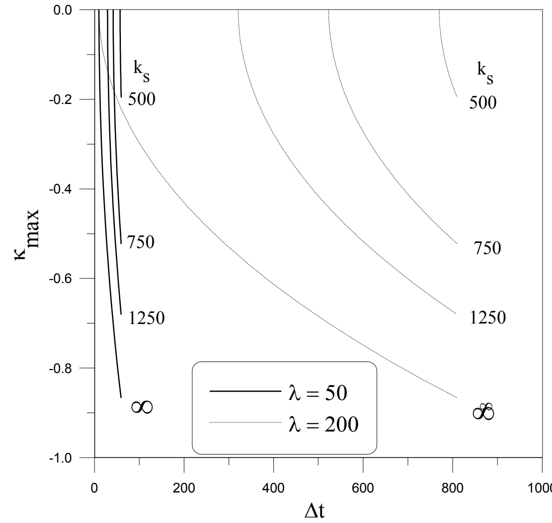

Figure 6d. Maximum Curvature.

Figure 7 presents the rod configurations for a total deformation $\varepsilon = 2\%$ and four spring stiffness $k_s =$ 500, 750, 1250 and ∞. The required temperatures for slenderness ratios $\lambda =$ 50 and 200 are also displayed.

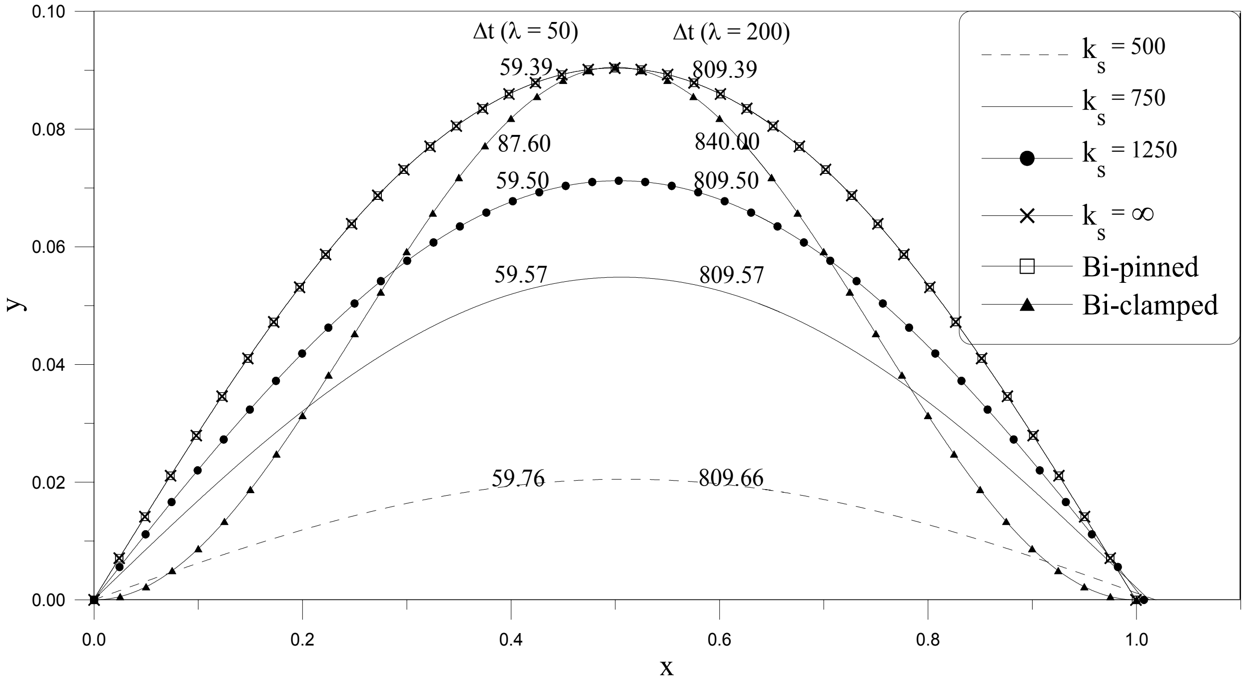

Figure 7. Deflected Configurations for Total Strain $\varepsilon = 2\%$.

6. Conclusion

The mathematical formulations, analytical and numerical solutions presented in this paper have been successfully employed in a two-point boundary value problem governed by a set of six first-order non-linear ordinary equations. The post-buckling analysis of slender elastic rods subjected to uniform temperature variation is highly dependent on the prescribed ends conditions. The rod boundary conditions are considered (i) double-hinged non-movable ends, (ii) hinged at the ends, being non-movable at one edge and axially restrained by a linear spring at the other edge and (iii) double-fixed non-movable ends.

The governing equations are written in non-dimensional form and it is noted that three parameters control the solution: slenderness ratio and spring stiffness. The critical buckling displacement, load and temperature are calculated and a parametric study is performed. The closed-form analytical solution via uncoupled elliptical integrals derived from the governing equations in the deflected configuration and the numerical technique was developed allowing a complete parametric study.

Results, presented in non-dimensional graphs, were obtained as a function of the slenderness ratios, temperatures and different values of the spring stiffness. The simple but powerful numerical procedure employed may facilitate further developments in two-point boundary value problems such as post-buckling analysis of slender elastic rods subjected to non-uniform thermal load and is the object of research.

Petrobras has taken advantage of several projects to investigate the behavior of heated pipelines. This work established an analytical tool that will serve as the basis for buckling analysis in new pipeline design.

7. Acknowledgement

The authors acknowledge the support from the National Petroleum Agency (ANP) and the National Council of Scientific and Technological Development (CNPq) for this work. The authors would like to thank also PETROBRAS for allowing the main author to develop this paper.

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