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A SOLUTION THROUGH INTEGRAL TRANSFORMS FOR FULLY DEVELOPED FLOW IN DOUBLY CONNECTED DUCTS

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Abstract. *A solution based on the Generalized Integral Transform Technique (GITT) is obtained for fully developed laminar flow of Newtonian fluids inside doubly connected ducts. The mathematical formulation is constructed based on the cylindrical coordinates system in such a way that the solid surfaces are described in terms of internal and external radii as functions of the angular coordinate, thus avoiding discontinuities in the boundary conditions. Three cases of doubly connected ducts are considered, namely, confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts. Numerical results for the velocity field and the product of the Fanning friction factor-Reynolds number were produced for different values of the governing parameters, according to the specific duct cross-section. The results were confronted with previously reported ones, providing critical comparisons while illustrating the employed integral transform approach.*

Keywords. *Doubly connected ducts, Channel flow, Integral transforms, Friction factor.*

1. Introduction

Laminar flow in annular passages is frequently found in a wide range of engineering and industrial applications such as in heat exchange devices, oil and gas drilling wells and extrusion processes, to name a few. Shah and London (1978) pointed out a number of works that dealt with this type of flow inside various doubly connected ducts. Among them, it can be mentioned the confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts, as the most relevant and frequently considered geometric configurations, mainly due their wide use in compact heat exchangers.

The compilation of Shah and London (1978) provided extensive information on fully developed laminar flow of Newtonian fluids in ducts with doubly connected cross-sections. A variety of different methods has been employed in the literature to obtain solutions for the governing differential equations and associated boundary conditions, most based on purely numerical techniques. Among the more analytically oriented contributions, one may cite the pioneering works of Piercy *et al.* (1933), Sastry (1965a, 1965b), Shivakumar (1973) that employed conformal mapping methods, and Topakoglu and Arnas (1974), which used an elliptical coordinates system to analyze the flow in confocal elliptical ducts. Solutions in closed-form were obtained by these authors for the velocity field and related flow characteristics. Shivakumar (1973) also analyzed the flow in elliptical ducts with central circular cores through conformal mapping. Attention has also been devoted to the analysis of flow and heat transfer in eccentric annular ducts. A literature review brings up some of the earlier studies on such ducts, which are attributed to Piercy *et al.* (1933), Stevenson (1949), Snyder and Goldstein (1965) and Jonsson and Sparrow (1965) that concentrated their analyses in the fluid flow, while Cheng and Hwang (1968), Trombetta (1971) and Suzuki *et al.* (1991) analyzed the heat transfer problem under different sets of boundary conditions. More recently, these studies have regained interest in the works of Manglik and Fang (1995), Fang *et al.* (1999), Manglik and Fang (2002) and Escudier *et al.* (2002) in which the effects of eccentricity and duct rotation were investigated for the flow and heat transfer of non-Newtonian fluids. In addition, the recent work by Escudier *et al.* (2002) offers a literature review for flow and heat transfer in eccentric annular ducts involving Newtonian and non-Newtonian fluids.

On the other hand, a hybrid analytical-numerical approach has been advanced for the solution of elliptic diffusion-type problems defined within irregular domains (Aparecido *et al.*, 1989), and applied to the analysis of fully developed laminar flow within ducts of various shapes, such as trapezoidal, triangular, and hexagonal ducts (Aparecido and Cotta, 1987; Aparecido *et al.*, 1989; Aparecido and Cotta, 1990; Barbuto and Cotta, 1997), by extending the ideas in the well-established Generalized Integral Transform Technique (GITT), as reviewed by Cotta (1993, 1994). Fully developed laminar flow and heat transfer of non-Newtonian fluids inside irregular ducts of different geometric configurations was also treated (Chaves *et al.*, 2001a; 2001b; 2004), again through extension of the GITT approach, yielding accurate numerical results for quantities of practical interest such as the Fanning friction factor and Nusselt numbers, within a wide range of the governing parameters.

Following this same line of research, the purpose of the present study is to solve the momentum equations for laminar fully developed flow in doubly connected ducts by employing the GITT approach, and to establish reliable numerical results for the velocity field and the Fanning friction factor, for different values of the governing parameters according to the specific cross-section under consideration. The cylindrical coordinates system is used in the mathematical formulation, so that the solid surfaces are described in the form of internal and external radii as functions of the angular coordinate. Three cases of doubly connected ducts are considered to illustrate the approach, namely, confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts. The first two cases are studied in order to demonstrate the ability of the present approach in handling such irregular domains. Then, the case of eccentric annular ducts is more closely analyzed, providing sets of benchmark results which were critically compared with those previously reported in the literature.

2. Analysis

One considers fully developed laminar flow in the annular passage of a general doubly connected duct as described in Fig. (1). The cylindrical coordinates system is used to map the duct, so that that the solid surfaces are described in terms of radii values, as functions of the angular coordinate, thus avoiding discontinuities in the boundary conditions. In addition, it is taken into account that the flow is steady-state and incompressible with constant properties. Impermeability and no-slip conditions at the duct walls are also considered. Then, the mathematical formulation of this problem is given by the momentum conservation equation, in the Z direction, as follows:

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial V_Z(R,\theta)}{\partial R}\right]+\frac{1}{R^2}\frac{\partial^2 V_Z(R,\theta)}{\partial \theta^2}=-C\,,\quad \text{in}\qquad R_1(\theta)<R<R_2(\theta);\quad 0<\theta<2\pi \tag{1}$$

$$V_Z(R_1(\theta),\theta)=0;\quad V_Z(R_2(\theta),\theta)=0 \tag{2,3}$$

$$V_Z(R,0)=V_Z(R,2\pi);\quad \frac{\partial V_Z(R,0)}{\partial\theta}=\frac{\partial V_Z(R,2\pi)}{\partial\theta} \tag{4,5}$$

The following dimensionless variables were employed in the above formulation:

$$R=\frac{r}{L_2},\quad V_Z(R,\theta)=\frac{v_z(r,\theta)}{\left(-\frac{dp}{dz}\right)\frac{D_h^2}{\mu}},\quad C=\left(\frac{L_2}{D_h}\right)^2,\quad R_1(\theta)=\frac{r_1(\theta)}{L_2},\quad R_2(\theta)=\frac{r_2(\theta)}{L_2} \tag{6}$$

where L_2 is a radial characteristic length of the outer surface, D_h is the hydraulic diameter of the duct and $r_1(\theta)$ and $r_2(\theta)$ are the functions that describe the inner and outer surfaces, respectively.

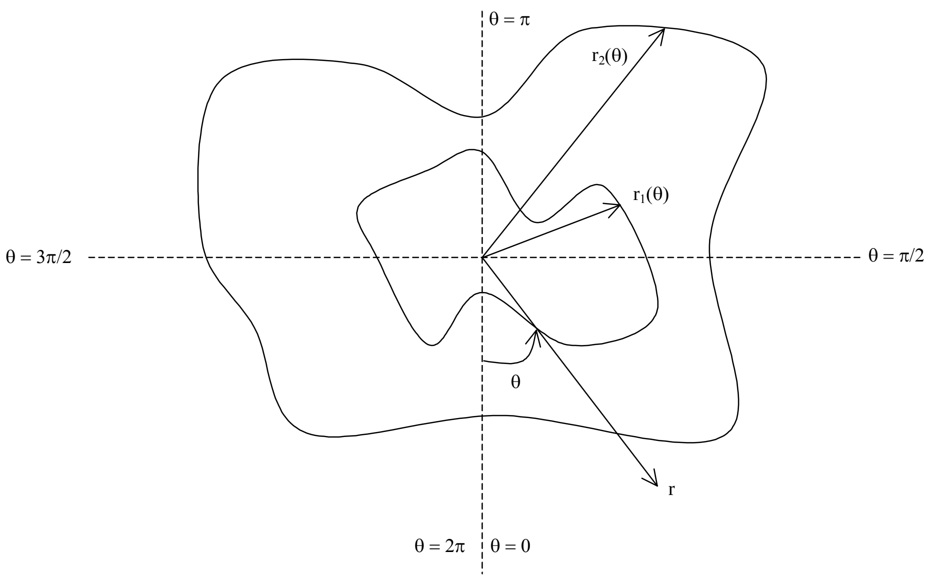

Figure 1. Coordinates system and geometrical representation of the general doubly connected duct.

2.1 Solution methodology

An analytical solution for the problem defined by Eqs. (1) to (5) is difficult to find in light of the irregular nature of the domain, given by the functions $R_1(\theta)$ and $R_2(\theta)$. In order to overcome this difficulty, the GITT approach will be employed in the hybrid numerical-analytical solution of the present formulation.

Following the ideas in the GITT (Cotta, 1993; 1994), an appropriate auxiliary eigenvalue problem is selected, which shall provide the basis for the eigenfunction expansion. Here, due to the angular dependence of the boundaries, it is needed to choose the radial coordinate R to provide the auxiliary eigenvalue problem and to be eliminated in the integral transformation process. Therefore, the following eigenvalue problem is proposed:

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\psi_i(R,\theta)}{\partial R}\right]+\mu_i^2(\theta)\frac{1}{R^2}\psi_i(R,\theta)=0\,, \quad \text{in} \qquad R_1(\theta)<R<R_2(\theta) \tag{7}$$

$$\psi_i(R_1(\theta),\theta)=0\,; \quad \psi_i(R_2(\theta),\theta)=0 \tag{8,9}$$

Equations (7) to (9) can be analytically solved to yield, respectively, the eigenfunctions and eigenvalues as

$$\psi_i(R,\theta)=\sin\left[\mu_i(\theta)\ln\left(\frac{R_2(\theta)}{R}\right)\right]; \quad \mu_i(\theta)=\frac{i\pi}{\ln\left(\dfrac{R_2(\theta)}{R_1(\theta)}\right)}\,, \quad i=1,2,3,... \tag{10,11}$$

It can be shown that the eigenfunctions $\psi_i(R,\theta)$ enjoy the following orthogonality property:

$$\int_{R_1(\theta)}^{R_2(\theta)}\frac{\psi_i(R,\theta)\psi_j(R,\theta)}{R}dR=\begin{cases}0, & i\neq j\\ N_i(\theta), & i=j\end{cases} \tag{12,13}$$

where $N_i(\theta)$ is the normalization integral computed as

$$N_i(\theta)=\int_{R_1(\theta)}^{R_2(\theta)}\frac{\psi_i^2(R,\theta)}{R}dR=\frac{1}{2}\ln\left(\frac{R_2(\theta)}{R_1(\theta)}\right) \tag{14}$$

Equations (7) to (9) together with the above properties allow the definition of the integral transform pair for the velocity field as:

$$\bar{V}_{Z,i}(\theta)=\frac{1}{N_i(\theta)}\int_{R_1(\theta)}^{R_2(\theta)}\frac{\psi_i(R,\theta)V_Z(R,\theta)}{R}dR\,, \qquad \text{transform} \tag{15}$$

$$V_Z(R,\theta)=\sum_{i=1}^{\infty}\psi_i(R,\theta)\bar{V}_{Z,i}(\theta)\,, \qquad \text{inverse} \tag{16}$$

To obtain the resulting system of differential equations for the transformed potentials $\bar{V}_{Z,i}(\theta)$, the partial differential equation (1) is multiplied by $\psi_i(R,\theta)/R$, integrated over the domain $[R_1(\theta),R_2(\theta)]$ in the R-direction, and the inverse formula, Eq. (16), is employed in place of the velocity distribution $V_Z(R,\theta)$, resulting in the following transformed ordinary differential system:

$$\frac{d^2\bar{V}_{Z,i}(\theta)}{d\theta^2}+\sum_{j=1}^{\infty}A_{ij}(\theta)\frac{d\bar{V}_{Z,j}(\theta)}{d\theta}+\sum_{j=1}^{\infty}B_{ij}(\theta)\bar{V}_{Z,j}(\theta)=C_i(\theta)\,, \quad i=1,2,3,... \tag{17}$$

The same operation can be performed over the θ-direction boundary conditions given by Eqs. (4) and (5), to furnish

$$\bar{V}_{Z,i}(0)=\bar{V}_{Z,i}(2\pi)\,; \quad \frac{d\bar{V}_{Z,i}(0)}{d\theta}=\frac{d\bar{V}_{Z,i}(2\pi)}{d\theta} \tag{18,19}$$

where the coefficients in Eq. (17) are defined as follows:

$$A_{ij}(\theta) = \frac{2}{N_i(\theta)} \int_{R_1(\theta)}^{R_2(\theta)} \frac{1}{R} \psi_i(R,\theta) \frac{\partial \psi_j(R,\theta)}{\partial \theta} dR \; ; \quad B_{ij}(\theta) = \frac{1}{N_i(\theta)} \int_{R_1(\theta)}^{R_2(\theta)} \frac{1}{R} \psi_i(R,\theta) \frac{\partial^2 \psi_j(R,\theta)}{\partial \theta^2} dR - \mu_i^2(\theta)\delta_{ij} \tag{20,21}$$

$$C_i(\theta) = -\frac{C}{N_i(\theta)} \int_{R_1(\theta)}^{R_2(\theta)} R\psi_i(R,\theta) dR \; ; \quad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{22,23}$$

The coefficients $A_{ij}(\theta)$, $B_{ij}(\theta)$ and $C_i(\theta)$ vary along θ, due to the irregular characteristic of the duct in this direction, and need to be reevaluated along the solution procedure. Equations (17) to (19) form an infinite nonlinear boundary value problem, which has to be truncated in a sufficiently high order NV, in order to compute the transformed potentials for the velocity field, $\bar{V}_{Z,i}(\theta)$, to within an user prescribed accuracy goal. For the solution of such a system, due to its expected stiff characteristics, specialized subroutines have to be employed, such as the subroutine DBVPFD from the IMSL Library (1991). This subroutine provides the important feature of automatic controlling the relative error over the solution of the ordinary differential equations system, allowing the user to establish error targets for the transformed potentials. Once this system is solved for the transformed potentials, the inverse formula, Eq. (16), is recalled to provide the velocity field.

In the realm of applications, related to pumping power estimation, one is concerned with quantities of practical interest such as the product of the Fanning friction factor-Reynolds number, fRe. As a consequence, the average flow velocity is also required, which in dimensionless form can be obtained as:

$$V_{Z,m} = \frac{v_{z,m}}{\left(-\dfrac{dp}{dz}\right)\dfrac{D_h^2}{\mu}} = \frac{\displaystyle\int_0^{2\pi} \int_{R_1(\theta)}^{R_2(\theta)} RV_Z(R,\theta) dR d\theta}{\displaystyle\int_0^{2\pi} \int_{R_1(\theta)}^{R_2(\theta)} R dR d\theta} = \frac{1}{A_t(\gamma)} \int_0^{2\pi} \int_{R_1(\theta)}^{R_2(\theta)} RV_Z(R,\theta) dR d\theta \tag{24}$$

where $A_t(\gamma)$ is the cross-section area of the annular passage and it is a function of the aspect ratio $\gamma = L_1/L_2$. Here L_1 is a radial characteristic length of the inner surface.

Applying the inverse formula, Eq. (16), into Eq. (24), the dimensionless average flow velocity becomes

$$V_{Z,m} = \frac{1}{A_t(\gamma)} \sum_{i=1}^{\infty} D_i \; ; \quad D_i = \int_0^{2\pi} E_i(\theta) \bar{V}_{Z,i}(\theta) d\theta \; ; \quad E_i(\theta) = \int_{R_1(\theta)}^{R_2(\theta)} R\psi_i(R,\theta) dR \tag{25-27}$$

The Fanning friction factor and Reynolds number are defined as:

$$f = \left(-\frac{dp}{dz}\right) \frac{(D_h/4)}{\rho v_{z,m}^2/2} \; ; \quad Re = \frac{\rho v_{z,m} D_h}{\mu} \tag{28,29}$$

Then, it is concluded that the product fRe is given as:

$$f Re = \frac{1}{2V_{Z,m}} \tag{30}$$

Three types of doubly connected ducts are considered in the present analysis, namely, confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts, as shown in Fig. (2). In order to improve the computational performance, since in all the cases there is symmetry in the velocity field at angular positions $\theta = 0$ and $\theta = \pi$, the domain in this direction is taken $[0,\pi]$, and the boundary conditions given by Eqs. (4) and (5) are replaced by

$$\frac{\partial V_Z(R,0)}{\partial \theta} = 0 \; ; \quad \frac{\partial V_Z(R,\pi)}{\partial \theta} = 0 \tag{31,32}$$

These conditions are also integral transformed following the steps previously described, to yield

$$\frac{d\bar{V}_{Z,i}(0)}{d\theta}=0\,;\quad \frac{d\bar{V}_{Z,i}(\pi)}{d\theta}=0 \tag{33,34}$$

The system given by Eq. (17) together with the transformed boundary conditions given by Eqs. (33) and (34) was then solved by the subroutine DBVPFD from the IMSL Library (1991). However, for general cases when there is no symmetry in the velocity field, one needs to consider the boundary conditions given by Eqs. (18) and (19).

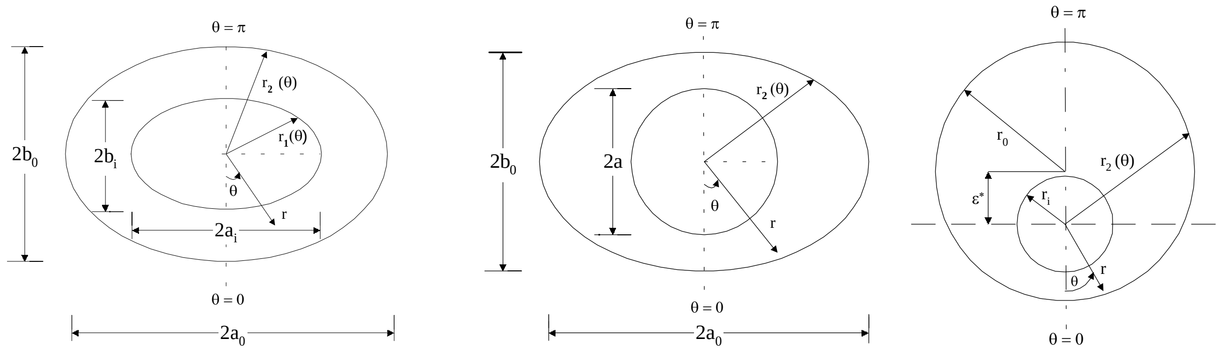

Figure 2. Details of the doubly connected ducts analyzed: confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts.

Table (1) below compiles information on the geometric configurations of the three cases of doubly connected ducts here considered. The additional parameters that appear in this table are: $\alpha = b_o/a_o$, the dimensionless relation of the semi-axes of the outer ellipse (for confocal elliptical ducts and elliptical ducts with central circular cores), $E(k_i^2)$ and $E(k_o^2)$, the complete elliptic integrals of the second kind, based on $k_i^2 = 1-\alpha^2\gamma^2/(\alpha^2\gamma^2+1-\alpha^2)$ and $k_o^2 = 1-\alpha^2$, respectively, and $\varepsilon = \varepsilon^*/(r_o - r_i)$, the dimensionless eccentricity of the annular duct.

Table 1. Geometric configurations of the confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts.

Parameter	confocal elliptical duct	elliptical duct with central circular core	eccentric annular duct
L_1	b_i	a	r_i
L_2	b_o	b_o	r_o
$R_1(\theta)$	$\gamma\sqrt{[\alpha^2\gamma^2+(1-\alpha^2)]/[\alpha^2\gamma^2+(1-\alpha^2)\cos^2\theta]}$	γ	γ
$R_2(\theta)$	$1\big/\sqrt{1-(1-\alpha^2)\sin^2\theta}$	$1\big/\sqrt{1-(1-\alpha^2)\sin^2\theta}$	$\sqrt{1-\varepsilon^2(1-\gamma)^2\sin^2\theta}-\varepsilon(1-\gamma)\cos\theta$
D_h	$\dfrac{\pi b_o(1-\gamma\sqrt{\alpha^2\gamma^2+1-\alpha^2})}{[E(k_i^2)\sqrt{\alpha^2\gamma^2+1-\alpha^2}+E(k_o^2)]}$	$\dfrac{2\pi b_o(1-\alpha\gamma^2)}{\pi\alpha\gamma+2E(k_o^2)}$	$2r_o(1-\gamma)$
$A_t(\gamma)$	$\pi b_o^2(1-\gamma\sqrt{\alpha^2\gamma^2+1-\alpha^2})\big/\alpha$	$\pi a_o b_o(1-\alpha\gamma^2)$	$\pi r_o^2(1-\gamma^2)$

3. Results and discussion

Numerical results for the product of the Fanning friction factor-Reynolds number and for the velocity field inside doubly connected ducts were obtained from a code developed in the FORTRAN 90 programming language. The code was implemented on a PENTIUM IV 1.7 GHz microcomputer, and the system given by Eqs. (17), (33) and (34) was handled through the subroutine DBVPFD from the IMSL Library (1991). A relative error target of 10^{-5} was employed throughout the computations. The complete solution was computed using up to twenty nine terms ($NV \le 29$) in the expansion, for all considered ducts, and the results were obtained in terms of geometric parameters α, γ and ε.

Tables (2) and (3) illustrate a comparison of the present results for the product fRe against those given by Shah and London (1978). The cases of confocal elliptical ducts and elliptical ducts with central circular cores are here analyzed in order to demonstrate the ability of the present approach in handling such irregular domains. As verified in Table (2) a reasonable overall agreement is obtained, to at least three significant digits between the two sets of results. This behavior provides a direct validation of the numerical code developed in the present work.

Table 2. Comparison of the product fRe for confocal elliptical ducts and elliptical ducts with central circular cores.

γ	α	fRe (confocal elliptical ducts)		γ	α	fRe (elliptical ducts with central circular cores)	
		Present work	Shah and London (1978)			Present work	Shah and London (1978)
0.5	0.6	21.536	21.585	0.5	0.5	19.321	19.321
0.5	0.9	23.678	23.773	0.5	0.7	21.694	21.694
0.6	0.4	20.273	20.171	0.5	0.9	23.520	23.519
0.6	0.9	23.746	23.819	0.6	0.9	23.435	23.435
0.7	0.6	21.826	21.896	0.7	0.7	19.403	19.402
0.7	0.9	23.804	23.851	0.7	0.9	23.159	23.159
0.95	0.9	23.895	23.896	0.95	0.9	16.822	16.816

Similarly, Table (3) brings a set of integral transform results for the case of eccentric annular ducts together with a comparison with those presented by Shah and London (1978). An excellent agreement is also verified for this type of duct in a wide range of governing parameters. In general, an increase in the aspect ratio γ, results in increasing fRe because of the reduction in the cross-section of the annular duct. On the other hand, the increase of ε diminishes the product fRe, basically due to an increase in the dimensionless average velocity.

Table 3. Comparison of the product fRe for eccentric annular ducts.

γ	fRe											
	ε = 0.05	ε = 0.1	ε = 0.2	ε = 0.3	ε = 0.4	ε = 0.5	ε = 0.6	ε = 0.7	ε = 0.8	ε = 0.9	ε = 0.95	ε = 1.0
0.005	19.505[a]	19.444	19.210	18.845	18.380	17.857	17.317	16.802	16.352	16.012	15.901	15.843
	19.504[b]	19.444	19.210	18.844	18.380	17.857	17.317	16.802	16.352	16.012	15.900	15.842
0.01	20.004	19.932	19.654	19.222	18.674	18.062	17.432	16.833	16.309	15.907	15.770	15.690
	20.004	19.932	19.654	19.221	18.674	18.061	17.432	16.833	16.309	15.907	15.769	15.690
0.02	20.600	20.512	20.174	19.651	18.993	18.262	17.514	16.805	16.182	15.693	15.517	15.400
	20.600	20.512	20.174	19.650	18.993	18.261	17.514	16.805	16.182	15.693	15.517	15.399
0.03	20.993	20.894	20.510	19.919	19.179	18.360	17.527	16.738	16.042	15.486	15.278	15.127
	20.993	20.893	20.509	19.918	19.179	18.360	17.527	16.737	16.041	15.486	15.277	15.126
0.04	21.290	21.180	20.759	20.111	19.304	18.414	17.511	16.656	15.900	15.288	15.051	14.870
	21.289	21.180	20.758	20.110	19.304	18.414	17.511	16.656	15.900	15.288	15.051	14.869
0.05	21.528	21.410	20.955	20.259	19.393	18.442	17.480	16.569	15.761	15.099	14.837	14.628
	21.528	21.409	20.955	20.258	19.393	18.442	17.479	16.569	15.761	15.099	14.836	14.627
0.06	21.727	21.601	21.117	20.377	19.459	18.454	17.439	16.479	15.626	14.919	14.633	14.399
	21.726	21.600	21.116	20.376	19.459	18.454	17.439	16.479	15.625	14.919	14.633	14.399
0.08	22.045	21.905	21.370	20.554	19.548	18.450	17.345	16.301	15.369	14.585	14.257	13.979
	22.044	21.905	21.369	20.553	19.548	18.450	17.345	16.301	15.369	14.584	14.257	13.978
0.10	22.292	22.141	21.562	20.681	19.599	18.423	17.243	16.129	15.131	14.280	13.918	13.601
	22.292	22.140	21.561	20.680	19.599	18.423	17.243	16.129	15.131	14.280	13.917	13.600
0.15	22.731	22.556	21.888	20.878	19.647	18.317	16.990	15.739	14.610	13.629	13.199	12.809
	22.731	22.555	21.887	20.877	19.647	18.317	16.990	15.739	14.610	13.629	13.198	12.808

a - Present work
b - Shah and London (1978)

Table 3. Continued.

γ	fRe											
	ε = 0.05	ε = 0.1	ε = 0.2	ε = 0.3	ε = 0.4	ε = 0.5	ε = 0.6	ε = 0.7	ε = 0.8	ε = 0.9	ε = 0.95	ε = 1.0
0.20	23.023[a]	22.830	22.094	20.986	19.641	18.197	16.760	15.407	14.181	13.105	12.625	12.185
	23.023[b]	22.829	22.093	20.985	19.641	18.197	16.760	15.406	14.181	13.105	12.625	12.184
0.25	23.232	23.024	22.234	21.049	19.615	18.081	16.558	15.125	13.825	12.677	12.161	11.684
	23.231	23.024	22.234	21.048	19.615	18.081	16.558	15.125	13.825	12.677	12.161	11.683
0.30	23.387	23.168	22.335	21.087	19.583	17.975	16.384	14.887	13.529	12.325	11.781	11.277
	23.387	23.168	22.335	21.086	19.582	17.975	16.384	14.887	13.528	12.325	11.781	11.276
0.40	23.598	23.362	22.465	21.125	19.515	17.800	16.107	14.517	13.073	11.791	11.210	10.670
	23.598	23.362	22.465	21.125	19.515	17.800	16.107	14.517	13.073	11.791	11.210	10.669
0.50	23.729	23.481	22.542	21.139	19.458	17.671	15.909	14.256	12.755	11.422	10.818	10.255
	23.729	23.481	22.541	21.139	19.458	17.671	15.909	14.256	12.755	11.422	10.818	10.254
0.60	23.811	23.555	22.587	21.144	19.415	17.579	15.770	14.075	12.537	11.170	10.551	9.9741
	23.811	23.555	22.587	21.144	19.415	17.579	15.770	14.075	12.537	11.170	10.551	9.973
0.70	23.861	23.601	22.615	21.146	19.386	17.518	15.678	13.955	12.392	11.004	10.375	9.7893
	23.861	23.601	22.615	21.145	19.386	17.518	15.678	13.955	12.392	11.004	10.375	9.788
0.80	23.891	23.628	22.631	21.146	19.367	17.480	15.622	13.882	12.304	10.903	10.268	9.6764
	23.891	23.627	22.631	21.145	19.367	17.480	15.622	13.882	12.304	10.903	10.268	9.675
0.90	23.906	23.642	22.639	21.145	19.358	17.460	15.593	13.844	12.258	10.850	10.213	9.6181
	23.908	23.642	22.639	21.145	19.358	17.460	15.593	13.843	12.258	10.850	10.213	9.617
1.00	23.910	23.645	22.642	21.145	19.355	17.455	15.584	13.833	12.245	10.835	10.197	9.6012
	23.910	23.645	22.642	21.145	19.355	17.455	15.584	13.833	12.245	10.835	10.196	9.600

a - Present work
b - Shah and London (1978)

Figures (3) show plots of the fully converged velocity ratio field at $\theta = 0$ and $\theta = \pi$ as function of the radial coordinate, $R_n = [R - R_1(\theta)]/[R_2(\theta)] - R_1(\theta)]$, for different eccentricities and aspect ratios. It is observed that an increase in the aspect ratio tends to make the velocity field symmetrical. The increase in the eccentricity causes an opposite effect at the two angular positions analyzed. For $\theta = 0$, as the eccentricity increases, the velocity peak tends to diminish; on the other hand, at $\theta = \pi$, the opposite behavior is verified, i.e., the velocity peak increases, which is explained by the continuity satisfaction, causing an overall increase of the average velocity. Figure (3b) also brings a comparison of the present results with those of Manglik and Fang (1995) for the aspect ratio $\gamma = 0.5$, and an excellent agreement of the results has been observed.

Figure (4) shows a comparison of the present results for the ratio between the maximum and the average velocity with those of Manglik and Fang (1995), at different angular positions, aspect ratios and eccentricities. Once again, it is noted a satisfactory agreement between the two sets of results. Also, one confirms that an increase in the eccentricity moves the velocity peak to the position $\theta = \pi$.

Figure (5) illustrates the effect of eccentricity in the isolines of the $V_Z/V_{Z,m}$ ratio. As expected, and previously discussed, the increase in the eccentricity tends to move the velocity peak to the angular position $\theta = \pi$. Finally, Fig. (6) shows the effect of aspect ratio on the isolines of the $V_Z/V_{Z,m}$ ratio. From this figure the symmetry in the velocity field is clearly observable for higher aspect ratios, since ducts with aspect ratios near unit tend to the configuration of a parallel-plates channel. It should also be noted that all the above figures do not show any oscillations in the eigenfunction expansions representations of the potential, over the entire solution domain.

4. Conclusions

A solution based on the GITT approach was developed to predict fully developed laminar flow of Newtonian fluids in doubly connected ducts, by considering three types of ducts for illustration, namely confocal elliptical ducts, elliptical ducts with central circular cores and eccentric annular ducts. The proposed integral transform approach provided reliable and cost effective simulations for the considered cases, by employing a cylindrical coordinates mapping of the solution domain and adopting an angular variable eigenvalue problem in the radial coordinate. Benchmark results for the product of the Fanning friction factor-Reynolds number and for the velocity field were systematically tabulated or graphically shown for different values of the governing geometric parameters, demonstrating the usefulness and robustness of the GITT alternative. The good agreement of the present results with those in the literature also served to demonstrate the ability of the integral transform approach in handling such class of problems, under the here introduced domain mapping and eigenfunction expansion representation. The effects of eccentricity and aspect ratio were also noticed in the product fRe, as well as in the resulting velocity field.

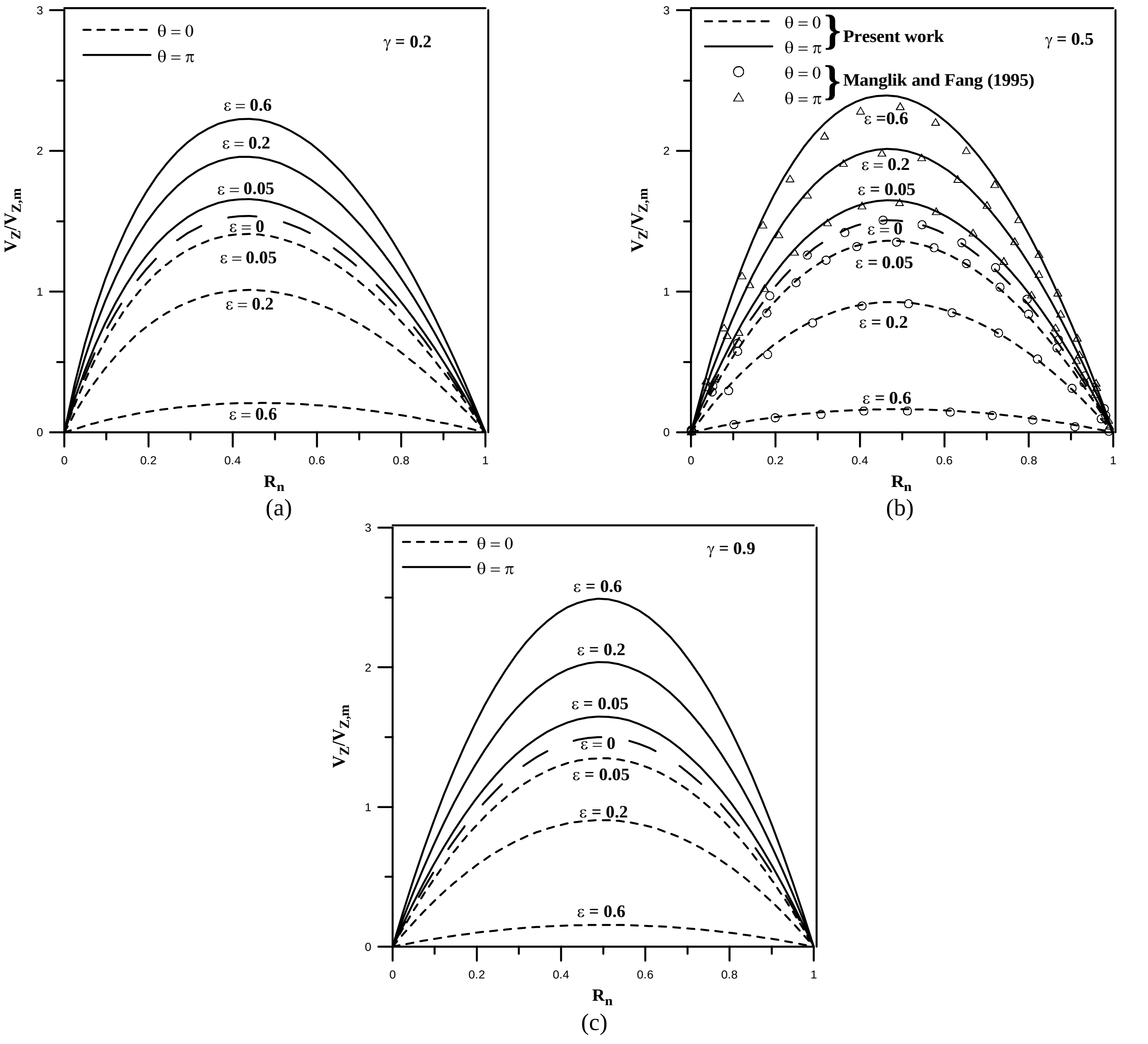

Figure 3. Behavior of the velocity ratio for eccentric annular ducts: (a) γ = 0.2; (b) γ = 0.5 and (c) γ = 0.9.

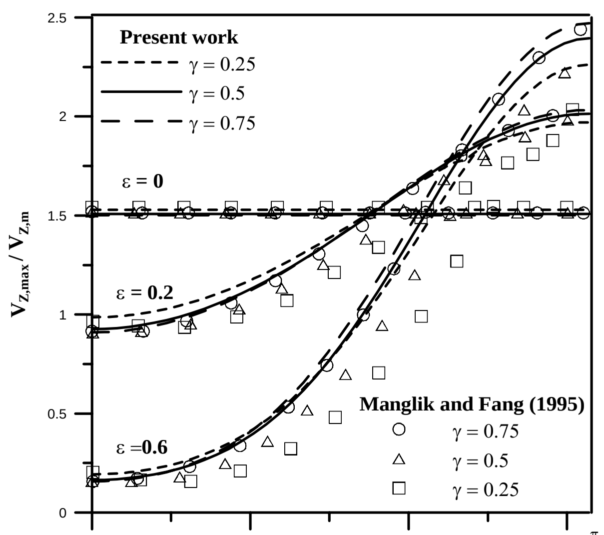

Figure 4. Comparison of the maximum to average velocity ratio for eccentric annular ducts.

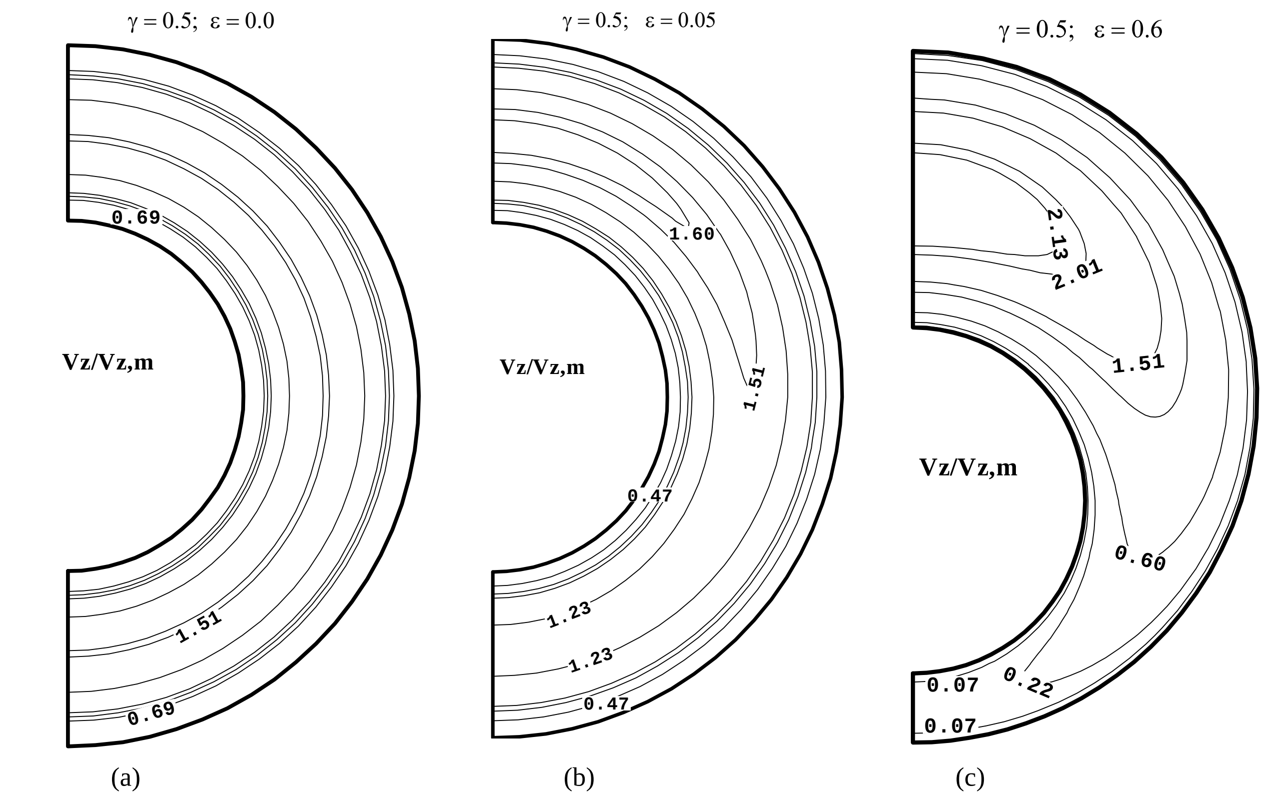

Figure 5. Effect of eccentricity on the isolines of the velocity ratio for eccentric annular ducts: (a) ε = 0.0 and γ = 0.5; (b) ε = 0.05 and γ = 0.5 and (c) ε = 0.6 and γ = 0.5.

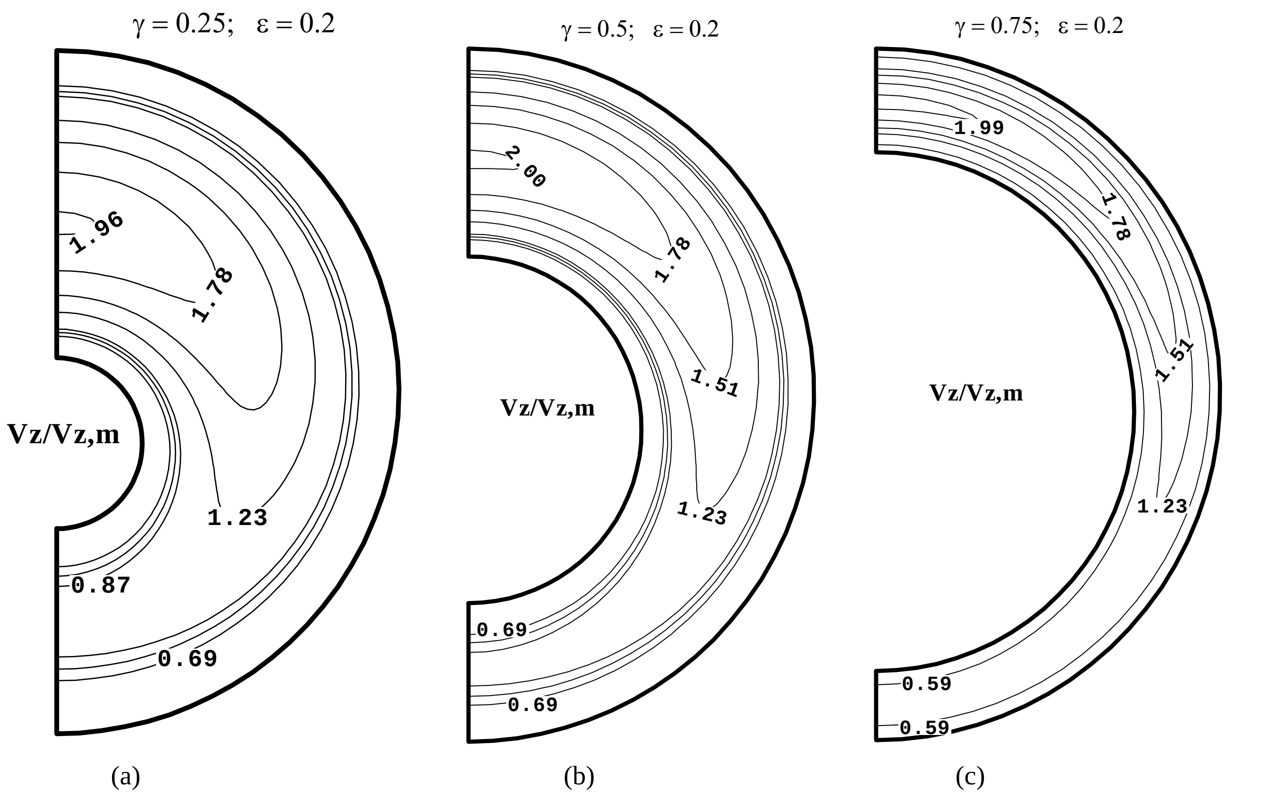

Figure 6. Effect of aspect ratio on the isolines of the velocity ratio for eccentric annular ducts: (a) γ = 0.25 and ε = 0.2; (b) γ = 0.5 and ε = 0.2 and (c) γ = 0.75 and ε = 0.2.

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