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ATMOSPHERIC BOUNDARY LAYER SIMPLIFICATIONS FOR FLOWS OVER LOW SLOPED HILLS

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Abstract. *The study of the atmospheric boundary layer flow over two-dimensional low-slope hills finds numerous applications in engineering and meteorology, such as the sitting of wind turbines, the estimation of wind loads on transmission towers and antennas and the development of computational programs to solve the atmospheric dynamics, called models. In this paper, a magnitude order analysis is applied to the momentum equations governing the airflow on the atmospheric boundary layer over low-slope hills. The method used is known as Prandtl's boundary layer simplifications, because it was originally used by Prandtl in 1904 to solve the flow over a flat plate. As a result of the analysis, simplified equations similar to Prandtl's are obtained. To the authors' knowledge, those equations are new in the context of atmospheric boundary layer studies. They cannot be solved analytically, but they still represent a great simplification to the complete equations of motion and, therefore, may be useful in numerical weather prediction schemes.*

Keywords: Atmospheric boundary layer, Prandtl´s boundary layer simplifications, micrometeorology.

1. Introduction

Topography is known to affect the Atmospheric Boundary Layer (ABL) flow in a number of ways. The most well known are the drag enhancement and the mean velocity speed-up. Also very important, however, are the modifications observed on the turbulence structure and, consequently, on the vertical diffusion of heat, momentum and scalar concentration. A simplified version of the general problem of airflow over complex terrain is the airflow over hills. It has been the subject of intense studies by engineers, meteorologists and environmentalists through the last decades. Studying the equations that govern the ABL flow over hills is the aim of the present paper.

The most important engineering application for the knowledge of airflow modifications induced by topography is probably the sitting of wind turbines in regions of enhanced wind speed, i.e., the top of the hills. Also very important is the estimation of wind loads on structures such as towers and antennas, which are preferably located on hilltops for obvious reasons. Other structures as chimneys and buildings are also often located on hilltops and constitute another important application. The modified wind pattern is also important information in the quantification of pollutant dispersal generated by industries located on hilly areas.

To meteorologists, the main application of flow over hills studies is in the development of computer programs to solve the atmospheric dynamics, called *models*. These models depend critically on the assumed velocity profile for the Atmospheric Boundary Layer. An equations describing the velocity distribution near the surface over flat terrain (the Logarithmic Law) has been widely used as a lower boundary condition during the integration of the governing equations in these models, in spite of the fact it is assumedly not valid for hilly terrain. The reason is that it is necessary to avoid the integration to be carried out all the way down to the surface, where strong velocity gradients would require a rather fine computational mesh. This application suggests that it is of great interest to have an expression for the velocity distribution in the airflow over hills, as recently proposed by Pellegrini and Bodstein (submitted). It also suggests that in case such an expression is not available, it would be useful to have simplified governing equations that could represent the wind flow near the surface. Other meteorological applications for such a relation are the direct modeling of u_* and the development of high-order turbulence closure schemes. Finnigan (1992) and Taylor and Lee (1984) list a number of other relevant meteorological and engineering applications to them in the engineering and atmospheric science. It is interesting to observe that, irrespective to the area, they could help improve our knowledge of the atmospheric flow over complex terrain.

Several descriptions of the flow over hills and complex terrain are available on the literature. An important part of this study was started with the seminal paper by Jackson and Hunt (1975) in which the flow field was divided into two sub-layers, each with different flow dynamics, using asymptotic expansion techniques. The validity of their analysis was then extended to three dimensions by Mason and Sykes (1979). Sykes (1980) also made other important contributions to the area showing (among other things) that it was necessary to include a third layer in order to consistently impose the surface boundary condition. Another contribution was given by Newley (1985), in the form of a detailed discussion on the various height scales of the problem. He also numerically modelled the problem and compared his results very favourably with field data (also on Belcher *et al.,* 1993). Two years latter, Zeman and Jensen (1987) solved the turbulence equations in streamline co-ordinates, establishing the role of each term and pointing out the importance of curvature effects.

The linear analyses initiated by Jackson and Hunt (1975) was further revised by Hunt *et al*. (1988a) and the result was the division of the flow field into four regions. The study of Hunt *et al*. (1988b) extended the results to stably stratified flow. The division proposed in these two papers has deeply influenced the understanding of the flow structure we have to the present days. Belcher (1990) refined the analysis even further, proposing a smooth match between the inner and outer regions.

An excellent review of the available studies on airflow over complex terrain is presented in Wood (2000), where numerical and observational studies are also included.

In this paper, an order of magnitude analysis is applied to the momentum equations governing the airflow on the atmospheric boundary layer over low-slope hills. The method used is known as Prandtl's boundary layer simplifications, because it was originally used by Prandtl (1904) to solve the flow over a flat plate. Following Finnigan (1983), the momentum equations are written in streamline coordinates. As a result of the analysis, simplified equations similar to Prandtl's are obtained. To the authors' knowledge, those equations are new in the context of atmospheric boundary layer studies. They cannot be solved analytically, but they still represent a great simplification to the complete equations of motion and, therefore, may be useful in numerical weather prediction schemes. The numerical implementation of the proposed set of equations in a neutrally stratified atmospheric model is currently under way. Some characteristics of the equations are analysed to conclude the paper. The proposed formulation is not complete due to the fact that no simplified energy equation and turbulence closure are present in this work.

2. Definition of the problem and governing equations

At the onset of the present analysis, it is important to realise that studies of airflow over hills should be considered as stepping-stones to the real problem that is airflow over complex terrain. It lies decades in the future the day when all the complexities of the real flow will be treated by a single theory. At the moment, the real problem is simplified in one of two ways: the terrain is considered as a succession of hills with a certain degree of symmetry (a sinusoidal wave, for example) or the hills are considered isolated and treated separately. The present study proceeds in the latter way.

The airflow around a hill is determined by its shape as well as by its height and length. In a great part of the works found in the literature, the former aspect is not considered. The reason is that any attempt to do so generally exceeds the sate of the art. Therefore, the shape of the hill is only indirectly acknowledged through the rate between its height and length. The present work purposes to take the real shape if the hill into consideration using streamline coordinates.

In the following study, a hill is defined following Kaimal and Finnigan (1994), as a topographical feature with characteristic length (to be defined below) less than 10 km. Greater features are considered mountains. The definition of low slope hill is a little bit unusual. For our purposes, a slope is considered low when no separation occurs. The definition is formulated this way because, as discussed in section 3, the present results do not apply to the separated region. According to Kaimal and Finnigan (1994), available data suggests that a critical downwind hill slope angle for separation is about 10^{o} in neutral atmosphere. This leads to hill heights of order of 1 km (according to our definition of hill) and occurs in wind flows over very rough two-dimensional ridges.

Under non-neutral conditions, the influence of the static stability of the atmosphere on the mean flow must be quantified. This can be done by a number of non-dimensional parameters, the most usual being the Grashof and Froude numbers (the latter only making sense if the atmosphere is stably stratified). In this work, static stability is considered through the Grashof number in section 3, where limits for the applicability of the proposed expressions are obtained. As for the critical downwind hill slope angle for separation in non-neutral atmosphere, the researched literature provides no suggested value. Therefore, the interaction between critical angle for separation and static stability is not studied here. However, the equations derived in this work are valid for wind flows over hills subject to the Grashof number restrictions derived in section 3 as long as separation does no occur, i.e., as long as the slope is kept low.

2.1. Definition of the problem

Consider an isolated two-dimensional low-sloped vegetated hill in the middle of an otherwise flat vegetated terrain of constant or slowly varying surface properties. Suppose a period of the day where the flow can be considered statistically stationary. Figure 1 illustrates the idealised flow. The vertical coordinate, z, is defined as the height above the local terrain, and the horizontal coordinate is x. The main geometrical parameters of the hill are its height, h, its surface curvature radius, R_h, and its horizontal length scale, L_h. The curvature radius is defined as usually in Calculus, as $(1+(dh/dx)^2)^{3/2} / |d^2h/dx^2|$. It can be physically interpreted as the radius of a circle tangent to a given curve with the same local rate of variation of its direction along the curve. Figure 1 shows the local radius of curvature of the hilltop (HT). This definition implies that $R_h < 0$ at the HT. The length scale L_h is defined as the horizontal distance from the HT to the half-height point. The location upwind of the hilltop where the velocity profile is not perturbed by the presence of the hill is called the reference site (RS).

The vertical profile of the mean horizontal wind at RS, $\overline{u}_0(z)$, is considered to be essentially logarithmic and is given by the well-known logarithmic law. Over the hill, the mean velocity is given by

$$\overline{u}(x,z) = \overline{u}_0(z) + \Delta\overline{u}(x,z), \tag{1}$$

where the speed-up $\Delta\overline{u}$ is caused by topographic and surface property variations. The speed-up is positive at HT because the flow is accelerated to satisfy the continuity equation, and is negative somewhere at the upwind slope because of hill curvature effects. Dividing the speed-up by the RS velocity, the relative speed-up, ΔS, is obtained:

$$\Delta S(x,z) = \frac{\Delta\overline{u}(x,z)}{\overline{u}_0(x,z)} = \frac{\overline{u}(x,z)}{\overline{u}_0(x,z)} - 1 .$$

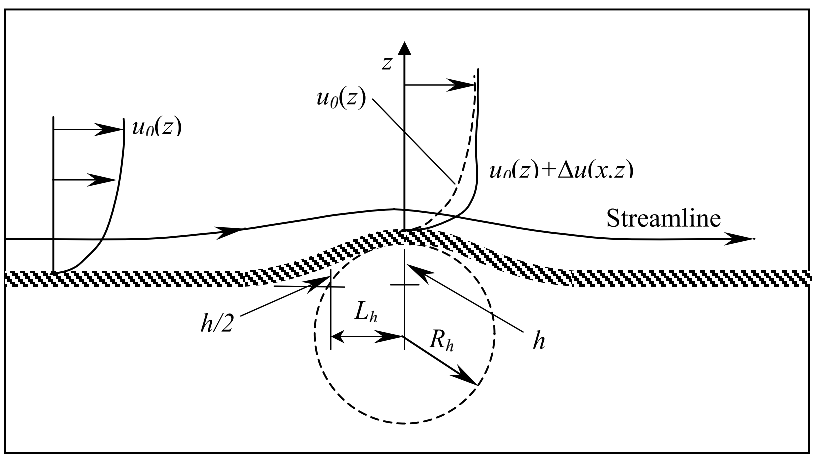

Fig.1 – Idealised flow over a low-slope hill

Over the last decades many works have been focused on determining the vertical profiles of $\Delta\overline{u}$ and ΔS, and also on calculating the maximum value of ΔS and the height where the maximum of $\Delta\overline{u}$ occurs. Except for Lemelin et al. (1988), most of the results are valid only for the HT, although it has been recognised that results applicable for the upwind and downwind slopes would be exceedingly valuable.

2.2. Governing equations for the ABL

There has been some agreement upon the fact that a curvilinear coordinate system is convenient to treat the ABL equations. Finnigan (1983) recommends the use of streamline coordinates and describes its advantages in the study of flow over hills. Following Finnigan (1983), the time-averaged governing equations for 2D, stationary flow in streamline coordinates under the Boussinesq approximation, can be written as

$$\frac{\overline{u}^2}{L_a} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} - \frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{u'w'}}{\partial z} + \frac{\overline{u'^2} - \overline{w'^2}}{L_a} + 2\frac{\overline{u'w'}}{R} - g_x\frac{\Delta \overline{T}}{T_0} + V_x \tag{3}$$

and

$$\frac{\overline{u}^2}{R} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} - \frac{\partial \overline{w'^2}}{\partial z} - \frac{\partial \overline{u'w'}}{\partial x} + \frac{\overline{u'^2} - \overline{w'^2}}{R} + 2\frac{\overline{u'w'}}{L_a} - g_z\frac{\Delta \overline{T}}{T_0} + V_z\ , \tag{4}$$

where the characteristic lengths, L_a and R, given by

$$\frac{1}{L_a} = \frac{1}{\overline{u}}\frac{\partial \overline{u}}{\partial x} \tag{5}$$

and

$$\frac{1}{R} = \frac{1}{\overline{u}}\left(\Omega + \frac{\partial \overline{u}}{\partial z}\right) \tag{6}$$

and the viscous terms by

$$V_x = \nu\left[\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial z^2} - \frac{2}{L_a}\frac{\partial \overline{u}}{\partial x} - \frac{1}{R}\frac{\partial \overline{u}}{\partial z} - \frac{\overline{u}}{R^2}\right] \tag{7}$$

$$V_z = \nu\left[-\frac{\partial^2 \overline{u}}{\partial x \partial z} + \frac{1}{L_a}\frac{\partial \overline{u}}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\overline{u}}{R}\right) + \frac{\overline{u}}{RL_a}\right] \tag{8}$$

In the preceding set of equations, x represents the direction parallel to the streamlines and $\overline{u}$ and u' the mean and turbulent velocities in the x direction, respectively. The direction normal to the streamlines is represented by z, and the corresponding turbulent velocity by w'. The thermodynamic mean pressure is denoted by $\overline{p}$ and the mean temperature by $\overline{T}$. Symbols T_0 and ρ_0 represent the reference temperature and density of the environment, respectively, considered hydrostatic with the air assumed to be an ideal gas. The difference between the mean temperature and the environmental temperature is denoted by $\Delta\overline{T}$. The components of gravity in the x and z directions are denoted by g_x and g_z, respectively, and the dynamic viscosity by ν.

In Eqs. (3) and (4), L_a can be interpreted as a length scale for the acceleration term in the x direction, whereas R can be interpreted as the local curvature radius of the streamlines in 2D flows. In Equation (6), Ω represents the mean component of vorticity in the direction normal to the plane of the flow. As Finnigan (1983) points out, Ω is an invariant of the transformation and therefore is written in the same way in both the streamline and the Cartesian coordinate systems. Finnigan *et al*. (1990) observes that 'both L_a and R are signed quantities, being positive if their local centres of curvature lie in the positive z and x directions, respectively'. For real hills, this means that $R < 0$ in the vicinity of the HT and $R > 0$ in the remaining parts of both slopes. It is interesting to notice that R and L_a are also properties of the velocity field. This is one of the most interesting differences between the streamline coordinates and more conventional systems as the terrain following coordinates: the geometric properties of the former are determined by the flow itself and not externally imposed.

In the set of Eqns. (3)–(8), the mass conservation equation is not included because it is automatically satisfied by the transformation. Finnigan (1992) shows that it must be substituted by the geometrical identity

$$\frac{\partial}{\partial x}\left(\frac{1}{L_a}\right) + \frac{\partial}{\partial z}\left(\frac{1}{R}\right) = \frac{1}{L_a^{\,2}} + \frac{1}{R^2} \tag{9}$$

A final interesting detail to be noticed on equations on Eqs. (3) and (4) is the fact that the simplification of the mean advection terms comes at the expense of extra turbulence terms. In fact, in both equations the advection terms were reduced from two to one, which represents advection along a streamline. At the same time although, turbulence terms as $2\overline{u'w'}/R$ were added to the right hand side of these equations as a consequence of the distorted coordinate system.

3. Magnitude order analysis

In this section we wish to derive the ABL analogue of Prandtl's (1904) boundary layer expressions valid for flat plates. Therefore, we begin with an overview of the method to be used. In its modern version (Schetz, 1993, for example) Prandtl's method consists in, first, making the governing equations of the problem non-dimensional. This must be done in such a way that every term of the equations is turned into a combination of order one variables multiplied by a non-dimensional parameter. Depending on the nature of the problem, important phenomena may occur very close to the boundaries of the flow, so that regions where the non-dimensional normal coordinate is nearly zero must be investigated. The next step therefore is to 'stretch' the non-dimensional normal coordinate through a variable transformation. This is achieved dividing the normal (non-dimensional) coordinate by a (non-dimensional) small parameter obtained during the preceding step. In Prandtl's (1904) work, the procedure was used to mathematically magnify the thin region of the flow field where friction forces are important. At this point, all terms of the equations are a combination of order one variables and non-dimensional terms and it is easy, therefore, to compare them and to eventually neglect the lowest order ones. After this simplification, the resulting equations can be transformed back to dimensional form and solved, if they are simple enough. Evidently, the choice of the parameters used to render the equations non-dimensional and to stretch the vertical coordinate is crucial in the method.

To perform the analysis described above, the notation proposed by Tennekes and Lumley (1972) is adopted. If the error involved is less than 30%, the symbol $\cong$ is used. Coarser approximations are represented by ~. This symbol is also used whenever two or more terms of an equation are compared. After the dominating terms have been established, the simpler notation = is used in the resulting equation, bearing in mind that the error can be made as small as necessary in some appropriate asymptotic limit. Whenever M is much greater (at least one order of magnitude larger) than N, the notation $M >> N$ is used.

To begin the analysis, Eqs. (3) and (4) are made non-dimensional by the following variable transformations: $X \equiv x/L_h$, $Z = z/L_h$, $U = \overline{u}/U_g$, $P = \overline{p}/(\rho_0 U_g{}^2)$ e $\Omega^{ad} = \Omega/U_g L_h$, where L_h is considered the horizontal length of the hill and U_g is the geostrophic wind speed. Although ABL problems usually do not consider Coriolis force effects, the geostrophic velocity is used because it represents an upper limit of the ABL velocity in most situations. The turbulence terms in Eqs. (3) and (4) can be made non-dimensional by the friction velocity, u_*. Dropping, for simplicity, the overbars representing the time average, the result is

$$\frac{U^2}{L_a{}^{ad}} = -\frac{\partial P}{\partial X} - \varepsilon_*{}^2\left(\frac{\partial U'^2}{\partial X} + \frac{\partial U'W'}{\partial Z} - \frac{U'^2 - W'^2}{L_a{}^{ad}} - 2\frac{U'W'}{R^{ad}}\right) - \varepsilon_R{}^2 \mathrm{Gr}_x + \varepsilon_R V_x{}^{ad} \quad (10)$$

and

$$\frac{U^2}{R^{ad}} = -\frac{\partial P}{\partial Z} - \varepsilon_*{}^2\left(\frac{\partial W'^2}{\partial Z} + \frac{\partial U'W'}{\partial X} - \frac{U'^2 - W'^2}{R^{ad}} - 2\frac{U'W'}{L_a{}^{ad}}\right) - \varepsilon_R{}^2 \mathrm{Gr}_z + \varepsilon_R V_z{}^{ad}\ , \quad (11)$$

where

$$\varepsilon_* = \frac{u_*}{U_g} \quad (12)$$

and

$$\varepsilon_R = \frac{1}{\mathrm{Re}} \quad (13)$$

are the small parameters. They are indeed small, since $u_* << U_g$ and $\mathrm{Re} >> 1$ in the ABL. In Eqs. (12) and (13), $\mathrm{Re} = U_g L_h/\nu$ is the Reynolds number and $\mathrm{Gr}_x = (g_x L_h^3/\nu^2)(\Delta\overline{T}/T_0)$ and $\mathrm{Gr}_z = (g_z L_h^3/\nu^2)(\Delta\overline{T}/T_0)$ are the Grashof numbers in the x and z directions, respectively. The variables $L_a{}^{ad}$ and R^{ad} are defined as expected as

$$\frac{1}{L_a{}^{ad}} = \frac{1}{U}\frac{\partial U}{\partial Z} \quad (14)$$

and

$$\frac{1}{R^{ad}} = \frac{1}{U}\left(\Omega^{ad} + \frac{\partial U}{\partial Z}\right). \tag{15}$$

Finally, the non-dimensional viscous terms are

$$V_x^{\,ad} = \left[\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2} - \frac{2}{L_a^{\,ad}}\frac{\partial U}{\partial X} - \frac{1}{R^{ad}}\frac{\partial U}{\partial Z} - \frac{U}{R^{ad\,2}}\right] \tag{16}$$

and

$$V_z^{\,ad} = \left[-\frac{\partial^2 U}{\partial X \partial Z} + \frac{1}{L_a^{\,ad}}\frac{\partial U}{\partial X} + \frac{\partial}{\partial X}\left(\frac{U}{R^{ad}}\right) + \frac{U}{R^{ad} L_a^{\,ad}}\right]. \tag{17}$$

The preceding equations could be simplified just considering that the atmospheric flow has typically very large Reynolds numbers. This would allow the viscous and turbulent terms to be neglected in Eqs. (10) and (11). Nevertheless, this simplification would come at a high price. As the viscous terms are neglected, the order of the partial differential equations falls from two to one, meaning that the no-slip boundary condition can no longer be satisfied. The result obtained with this kind of consideration is seldom useful in ABL studies. Furthermore, the transformation proposed to render *z* non-dimensional does not guarantee that $Z \square 1$, complicating the order of magnitude comparisons to come.

Both problems mentioned above were recognised by Prandtl in his original analysis. To avoid them, the author proposed the following transformation of the vertical coordinate:

$$Z^* = \frac{Z}{\varepsilon} = \frac{z}{\varepsilon L_h}, \tag{18}$$

where ε is a small parameter to be specified.

The idea behind Eq. (18) is that it is possible to choose an adequately small value to ε so as to mathematically ‘stretch’ the thin region next to the surface where the viscous terms cannot be neglected. To clarify how the method works, suppose as Jackson and Hunt (1975), Hunt *et al.* (1988b) and Kaimal and Finnigan (1994) do (for other purposes), the existence of a region where mean flow advection and cross-stream divergence of shearing stress balance each other in the ABL. Equation (11) with Eq. (14) substituted into it then yields $U\,\partial U/\partial X \sim \varepsilon_*^{\,2}(\partial U'W'/\partial Z)$. Recalling that all variables, except possibly *Z*, must have magnitude order one, the previous relation suggests that $Z \sim \varepsilon_*^{\,2}$. As $Z = z/L_h$, this implies that the balance between inertia and turbulence occurs in a region where $z \sim L_h \varepsilon_*^2 = L_h (u_*^2/U_g^2)$. Conversely, this shows that the existence of a region where advection and shearing stress balance could be accessed substituting a stretched variable $Z^* = Z/\varepsilon_*^2$ of order one on Eq. (11). Therefore, using Eq. (18) with $\varepsilon \sim \varepsilon_*^2$ and assuming that after the stretching $Z^* \sim 1$, implies that $Z \sim \varepsilon_*^{\,2}$. Substituting this conclusion in Eq. (12) readily yields $U\,\partial U/\partial X \sim \varepsilon_*^{\,2}(\partial U'W'/\partial Z)$ as before. From Eq. (18) and $Z^* \sim 1$, again it follows that $z \sim \varepsilon L$, meaning that advection and shearing stress balance in a region defined by $z \sim L_h \varepsilon_*^2 = L_h (u_*^2/U_g^2)$.

Returning to the problem, Eq. (18) is substituted into Eqs. (10) and (11) and the viscous terms and R^{ad} are written out. The result is

$$\begin{aligned}\frac{U^2}{L_a^{\,ad}} = -\frac{\partial P}{\partial X} - \varepsilon_*^{\,2}\left[\frac{\partial U'^2}{\partial X} + \frac{1}{\varepsilon}\frac{\partial U'W'}{\partial Z^*} - \frac{U'^2 - W'^2}{L_a^{\,ad}} - 2\frac{1}{\varepsilon}\frac{U'W'}{U}\left(\varepsilon\Omega^{ad} + \frac{\partial U}{\partial Z^*}\right)\right] \\ -\varepsilon_R^{\,2}\mathrm{Gr}_x + \varepsilon_R\left[\frac{\partial^2 U}{\partial X^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 U}{\partial Z^{*2}} - \frac{2}{L_a^{\,ad}}\frac{\partial U}{\partial X} - \frac{1}{\varepsilon^2 U}\left(\varepsilon\Omega^{ad} + \frac{\partial U}{\partial Z^*}\right) - \frac{1}{\varepsilon^2 U}\left(\varepsilon\Omega^{ad} + \frac{\partial U}{\partial Z^*}\right)^2\right]\end{aligned} \tag{19}$$

and

$$\frac{U^2}{L_a^{ad}} = -\frac{\partial P}{\partial X} - \varepsilon_*^2 \left[\frac{\partial U'^2}{\partial X} + \frac{1}{\varepsilon}\frac{\partial U'W'}{\partial Z^*} - \frac{U'^2 - W'^2}{L_a^{ad}} - 2\frac{1}{\varepsilon}\frac{U'W'}{U}\left(\varepsilon\Omega^{ad} + \frac{\partial U}{\partial Z^*} \right) \right] - \varepsilon_R^2 \mathrm{Gr}_x + \varepsilon_R \left[\frac{\partial^2 U}{\partial X^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 U}{\partial Z^{*2}} - \frac{2}{L_a^{ad}}\frac{\partial U}{\partial X} - \frac{1}{\varepsilon^2 U}\left(\varepsilon\Omega^{ad} + \frac{\partial U}{\partial Z^*} \right) - \frac{1}{\varepsilon^2 U}\left(\varepsilon\Omega^{ad} + \frac{\partial U}{\partial Z^*} \right)^2 \right] \quad (20)$$

The analysis of the preceding equations may be simplified with a change in notation. The advective term on the left-hand side of Eq. (19) may be denoted A_x. On the right-hand side, we denote the pressure term by P_x, the turbulence terms (second and third) by T_{x1} and T_{x2}, the curvature terms (fourth and fifth) by C_{x1}, and C_{x2}, the buoyancy term (sixth) by B_x and the viscous terms (the remaining) by $V_{x1}, \ldots, V_{x5}$. Analogous definitions are used for Eq. (20). After this change in notation and after multiplying Eqs. (19) and (20) throughout by ε^2 and ε, respectively, we obtain

$$\varepsilon^2 A_x = -\varepsilon^2 P_x - \varepsilon_*^2 \left(\varepsilon^2 T_{x1} + \varepsilon T_{x2} - \varepsilon^2 C_{x1} - \varepsilon C_{x2} \right) - \varepsilon^2 \varepsilon_R^2 B_x + \varepsilon_R \left(\varepsilon^2 V_{x1} + V_{x2} - \varepsilon^2 V_{x3} - V_{x4} + V_{x5} \right) \quad (21)$$

and

$$A_z = -P_z - \varepsilon_*^2 \left(T_{z1} + \varepsilon T_{z2} - C_{z1} - \varepsilon C_{z2} \right) - \varepsilon \varepsilon_R^2 B_z + \varepsilon_R \left(-V_{z1} + V_{z2} + V_{z3} + V_{z4} \right) \quad (22)$$

Following Prandtl's method, suppose now that there is a region where the largest viscous terms cannot be neglected in Eqn (21) in comparison with the advective term. This imposes that $\varepsilon^2 \sim \varepsilon_R$. If we extend this idea to the largest turbulent terms, one further restriction is imposed, namely $\varepsilon_*^2 \sim \varepsilon$. Substituting these two restrictions back into Eqns. (21) and (22) yields,

$$\varepsilon_R A_x = -\varepsilon_R P_x - \sqrt{\varepsilon_R} \left(\varepsilon_R T_{x1} + \sqrt{\varepsilon_R} T_{x2} - \varepsilon_R C_{x1} - \sqrt{\varepsilon_R} C_{x2} \right) - \varepsilon_R^3 B_x + \varepsilon_R \left(\varepsilon_R V_{x1} + V_{x2} - \varepsilon_R V_{x3} - V_{x4} + V_{x5} \right) \quad (23)$$

and

$$A_z = -P_z - \sqrt{\varepsilon_R} \left(T_{z1} + \sqrt{\varepsilon_R} T_{z2} - C_{z1} - \sqrt{\varepsilon_R} C_{z2} \right) - \varepsilon_R^{5/2} B_z + \varepsilon_R \left(-V_{z1} + V_{z2} + V_{z3} + V_{z4} \right) \quad (24)$$

We now have two restrictions over ε, namely $\varepsilon^2 \sim \varepsilon_R$ and $\varepsilon \sim \varepsilon_*$, that cannot be satisfied at the same time. This fact points out that at least two regions must be defined to adequately describe the boundary layer: one where turbulent effects are important and another where viscous effects are important. Most analyses found in literature fail to acknowledge this fact, which is implicit in previous works of Pellegrini (2001) and Pellegrini and Bodstein (2002 and submitted). Even tough, the approximation proposed here is as useful as Prandtl's (1904) simplification.

Keeping only first order terms in Eqns. (23) and (24) results in

$$A_x = -P_x - \left(T_{x2} - C_{x2} \right) - \varepsilon_R^2 B_x + \left(V_{x2} - V_{x4} + V_{x5} \right) \quad (25)$$

and

$$A_z = -P_z - \varepsilon_R^{5/2} B_z \quad (26)$$

Keeping the buoyancy terms in the preceding equations deserves some attention. Going back to Eq. (19), we see that $B_x = \mathrm{Gr}_x = (g_x L_h^3 / \nu^2)(\Delta\overline{T}/T_0)$. Consequently, B_x cannot be calculated in the scope of the present analysis because it depends on the vertical distribution of temperature, implicit in $\Delta\overline{T}$, which can only be calculated using the energy conservation equation, which depends on the static state of the atmosphere. On the lack of this information, B_x must be considered an external condition (or forcing, in the meteorological language) to the problem, which must then be known *a priori* if the question about keeping or not B_x in the simplified Eq. (19) is to be decided. As all terms in Eqs. (25) and (26) have order one, it is obvious that for the buoyancy term to be considered, one must have

$$\varepsilon_R^2 B_x \sim 1 \quad (27)$$

and

$$\varepsilon_R^{5/2} B_z \sim 1 . \quad (28)$$

It is also obvious that if $\varepsilon_R B_x \sim 1$ and $\varepsilon_R^{5/2} B_z \sim 1$, the buoyancy terms can be neglected. However, if $\varepsilon_R B_x \sim 1$ and $\varepsilon_R^{5/2} B_z \sim 1$, these terms are the only remaining in Eqs. (25) and (26) and are said to dominate them. In such cases, as buoyancy forces are very strong, the atmosphere is evidently strongly diabatic and the Boussinesq approximation used to obtain Eqs. (3) and (4) fails as variations in ρ cannot be further neglected. Consequently, Eqs. (3) and (4) are no longer valid. The conservation of energy equation must also be considered in order to promote the necessary coupling between the velocity and temperature fields, case in which B_x ceases to be an external forcing.

Equations (25) and (26), subjected to their respective constraints, can be returned to their dimensional form by substituting the definitions of the various terms and the non-dimensional variables, parameters and numbers back into them. The result is

$$\frac{\overline{u}^2}{L_a} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} - \frac{\partial \overline{u'w'}}{\partial z} + 2\frac{\overline{u'w'}}{R} - g_x \frac{\Delta \overline{T}}{T_0} + \nu\left[\frac{\partial^2 \overline{u}}{\partial z^2} - \frac{1}{R}\frac{\partial \overline{u}}{\partial z} - \frac{\overline{u}}{R^2}\right] \tag{29}$$

and

$$\frac{\overline{u}^2}{R} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} - g_z \frac{\Delta \overline{T}}{T_0}, \tag{30}$$

valid for $z/L_h \sim (u_*/U_g)^2$. This single restriction corresponds to $\varepsilon \sim \varepsilon_*$, which represents the larger limit established for *z* in the stretching process. It was chosen that way because otherwise the influence of the turbulent terms would be neglected. The restrictions to be fulfilled in order to keep the buoyancy terms are

$$\mathrm{Gr}_x/\mathrm{Re}^2 \sim 1 \tag{31}$$

for the *x*-momentum equation and

$$\mathrm{Gr}_z/\mathrm{Re}^{5/2} \sim 1 \tag{32}$$

for the *z*-momentum equation with $\mathrm{Gr}_x = (g_x L_h^3/\nu^2)(\Delta\overline{T}/T_0)$, $\mathrm{Gr}_z = (g_z L_h^3/\nu^2)(\Delta\overline{T}/T_0)$ and $\mathrm{Re} = U_g L_h/\nu$. If $\mathrm{Gr}_x/\mathrm{Re}^2 \sim 1$ or $\mathrm{Gr}_z/\mathrm{Re}^{5/2} \sim 1$, the buoyancy terms can be neglected in their respective equations and if $\mathrm{Gr}_x/\mathrm{Re}^2 \sim 1$ or $\mathrm{Gr}_z/\mathrm{Re}^{5/2} \sim 1$, the energy conservation equation must be taken into account.

4. **Conclusions**

In the present work, Prandtl's method was used to simplify the ABL momentum equations for the airflow over low-slope hills. To the authors' knowledge, the resulting equations (Eqns. (29) and (30)) are new in the context of ABL studies. These equations cannot be solved analytically, but they still represent a great simplification to the original equations (Eqs. (3) and (4)). The simplified equations obtained here can be useful in making atmospheric mesoscale models give better results. Such kind of implementation is currently under way. The model chosen for the task was the Pennsylvania State University and NCAR model, the MM5. As the ability of the MM5 model to simulate microscale processes is somewhat limited, it is proposed that Eqns. (29) and (30) are used following Taylor (1998). In his work, Taylor suggests that site-specific wind forecasts could be made using formulations like Eqns. (29) and (30) and the predictions from larger scale models as input in a kind of nesting. To implement this idea, it is necessary first to write a microscale model able to solve Eqns. (29) and (30). The present study constitutes, therefore, the first step of a work in progress.

Also under way is a study to verify if the present simplified equations may be further simplified by asymptotic, perturbation or magnitude order methods. One possibility is to follow Pellegrini and Bodstein (2002, submitted) and use the Intermediate Variable Technique. If this idea proves mathematically coherent, the resulting equations may have analytical solution.

Comparing Eq. (29) to Eq. (3), we see that turbulent terms containing $\overline{u'^2}$ and $\overline{w'^2}$ and two out of the original five viscous terms were neglected. If, on one hand, the viscous terms are known not to bring many problems for the numerical solution, the turbulence terms, on the other, generally do. The simplification proposed on Eq. (3) implicates that two less prognostic equations for the turbulent terms are necessary. This fact suggests that more elaborate turbulence models can be used in the solution of a particular problem with the same computational effort. The same is true about Eq. (4), where all turbulent terms were neglected. Observing it we see that it is simply Euler's equation in a direction normal to the streamlines – all viscous terms were also neglected. This simplification is likely to be at least as good as considering the atmosphere hydrostatic, an operational option most atmospheric models use.

Some hypothesis were implicitly made during the deduction of Eqns. (29) and (30) and deserve some consideration. When the stretching process was used for the *z* direction only, it was assumed that no sudden changes in the dependent variables of the problem occurred in the *x* direction. This condition only holds if all external forcings vary slowly in the *x* direction. This is why the terrain conditions were supposed to be slowing varying at the onset of our analysis. For example, hills covered with grass but with forested patches left are not covered by the theory. Neither are hills with humps, depressions or buildings on the slopes.

The *x*-direction stretching only is also the reason why just low slope hills were considered. As the slope increases, the mean velocity gradients on both slopes also increase getting eventually to the point were separation occurs. The steep mean velocity gradients in the *x* direction are likely to spoil the approximations made here even before separation is reached. The fact that high atmospheric instabilities also lead to steep mean velocity gradients, implicates that no such cases are adequately modelled by Eqns. (29) and (30) either.

Another implicit hypothesis that make Eqns. (29) and (30) fail under strong instabilities and high slopes scenarios is the assumption that atmospheric eddies are relatively small and have comparable sizes in the *x* and *z* directions. This hypothesis was done when it was assumed that turbulence terms in Eqs. (3) and (4) could be made non-dimensional by the same friction velocity, u_*, which is much smaller that the geostrophic velocity, resulting that ε_* is in fact a small parameter. For ε_R to be also a small parameter, the analysis implicitly supposed that the Reynolds number is high. This is generally the case in atmospheric flows except for very low wind speeds and very small hills (remember that Re is defined using the characteristic length of the hill).

5. **References**

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