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Evolution Equation for Two-Phase Flow Hydrodynamic Instabilities in Pipe-Riser Systems.

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Abstract: We present a differential-algebraic model for gas-liquid flows in pipe-riser system which takes into account the pipe partial flooding with liquid and the gas cavity formation at the top of the riser that may occur during hydrodynamic instabilities, like severe slugging. We present an asymptotic theory that leads to the Landau equation as the evolution equation for the amplitude of the unstable modes that characterize the instabilities in this type of systems. As a by product, we obtain a stability criteria for steady gas-liquid flows in pipe-riser systems.

Keywords: Two-phase flows, Severe slugging, Hydrodynamic instability, Landau equation, Stability criteria.

INTRODUCTION

Under certain conditions, a steady two-phase flow with constant gas and liquid mass flow rates does not exists in pipe-riser systems, and intermittent flow regimes are observed. Whenever a sub-sea pipeline ends at a vertical riser connected to a platform separator, liquid may accumulate at the base of the riser and stop the gas motion. When this happens, the upstream gas is being compressed and its pressure rises while the liquid accumulates at both riser and pipeline. This situation continues until the gas pressure is large enough to push the liquid slug out of the pipeline and to start gas penetration into the riser. At this point different scenarios may happen. If the liquid slug pushed by the gas did not fill in the whole riser, gas bubbles penetrate into the riser, decreasing the pressure along the riser and making the gas-liquid mixture to flow out of the riser. The gas pressure in the pipeline decreases until the gas passage at the base of the riser is blocked again by the liquid and the repetitive cycle (limit cycle) starts again. If the liquid slug pushed by the gas has already filled in the riser, the separator has been only seeing liquid by the time the gas is able to push the liquid slug out of the pipeline. The gas pressure at the pipeline reached its maximum when bubble penetration into the riser had started. The bubble penetration stage is characterized by a rapidly expanding gas bubble that continuously overruns the riser liquid and leaves a thin film along the riser wall. This stage continues until the bubble enters the separator as a gas burst. This point is the start of the gas blowdown stage. During this stage, the gas inventory in the pipeline decreases steadily with a corresponding rapid decrease in pipeline pressure. At some point, the gas velocity in the riser becomes insufficient to support liquid on the riser wall, what marks the end of the gas blowdown stage and the beginning of the "liquid fallback" stage. During this stage, the gas passage is blocked,and a gas cavity is formed above the liquid accumulated at the bottom of the riser, and the repetitive cycle (limit cycle) starts again. The last of the instabilities described above is known as severe slugging phenomenon in the literature.

The cyclic behavior observed in the hydrodynamic instabilities in the pipe-riser systems suggests that the lost of stability is associated with a supercritical Hopf bifurcation, as reported in Zakarian (2000). According to the literature (see Aranha 2004), the Landau equation is a model equation for the scenario just described (i.e, lost of stability through a supercritical Hopf bifurcation leading to a limit cycle).

The objective of this work is to obtain the Landau equation from the governing equation for two-phase flows in a pipe-riser system. The Landau equation coefficients σ and μ are obtained directly from the governing equations. In the process of deriving the Landau equation as an evolution equation for the instabilities in the two-phase flows in pipe-riser systems, we also derived a stability criteria for steady two-phase flows in pipe-riser systems.

The model developed here is an improvement over the model presented in Zakarian (2000), since it takes into account the possibility of partial pipe flooding and the formation of gas cavity at the top of the riser, which happens in the case of severe slugging phenomenon.

In the next section, we present the model used for two-phase flows in a pipe-riser system and the associated governing equations in terms of non-dimensional variables. In the third section and in its subsections, we present the asymptotic analysis of the system non-dimensional governing equations, which leads to the Landau equation as the evolution equation for the pair of unstable modes associated with the Hopf Bifurcation. As a side product of the asymptotic analysis, a stability criteria for the steady gas-liquid flow in pipe-riser systems is presented in the fourth section. In the fifth section, we present our discussion and conclusions.

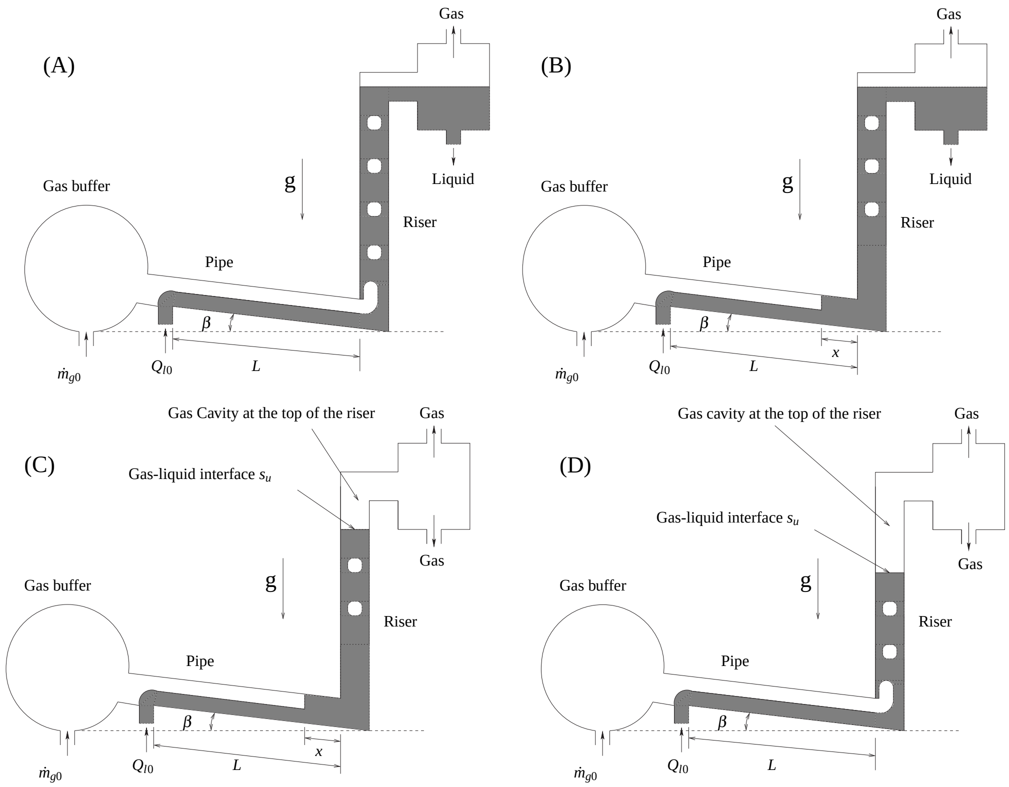

Figure 1 – Part (A) - First configuration: $x = 0$ and no gas cavity. Part (B) - Second configuration: $x > 0$ and no gas cavity. Part (C) - Third configuration: $x > 0$ and gas cavity. Part (D) - Fourth configuration: $x = 0$ and gas cavity.

TWO-PHASE FLOW MODEL.

The pipe-riser system is composed basically of two parts. The pipe plus a gas buffer and the riser. The pipe and riser are connected at the base of the riser. The pressure at the top of the riser is assumed to be the atmospheric pressure and we have liquid and gas mass flowing into the pipe (see figure 1).

The gas-liquid flow in the pipe is assumed as always stratified. This flow behavior extends either to the whole pipe (see parts (A) and (D) of figure 1) or it extends until the liquid penetration position in the pipe (see parts (B) and (C) of figure 1). The first configuration corresponds to continuous gas flow from the pipe into the riser and the second configuration corresponds to no gas flow from the pipe into the riser and partial liquid flooding of the pipe. Q_{l0}, $\dot{m}_{g0}$, β, L, g and x illustrated in figure 1 represents, respectively, the volumetric flow rate of liquid into the pipe, the gas mass flow rate into the pipe, the pipe inclination angle, the distance of the liquid inlet from the base of the riser, the gravity acceleration constant and the pipe liquid flooding distance from the base of the riser (parts (B) and (C) of figure 1).

We consider an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser. For the riser we consider two configurations. The riser is either filled with the gas-liquid mixture (see parts (A) and (B) of figure 1) or it may have an interface separating the gas-liquid mixture from a gas cavity located at the top of the riser. s_u represents, when it exists, the position of the liquid-gas interface in the riser with respect to the base of the riser.

Therefore, we have a total of four different configurations. The first (fourth) configuration is illustrated in part (A) (part (D)) of figure 1. In this configuration we have stratified flow in the pipe and continuous gas penetration from the pipe into the riser and the riser is totally (partially) filled with the gas-liquid mixture. The second (third) configuration is illustrated in part (B) (part (C)) of figure 1, where we have stratified flow in part of the pipe with liquid flooding until a distance x from the base of the riser, which is totally (partially) filled with the gas-liquid mixture. In the fourth and third configurations we have a gas cavity at the top of the riser.

The set of governing equations is not the same for the four different configurations represented in figure 1. Below we give governing equations for the different configurations illustrated in figure 1.

Governing Equations for the Pipe-riser System.

We give the governing equations for the pipe-riser system in non-dimensional form. We define the following non-dimensional variables according to the set of equations below.

$$x^* = \frac{x}{L_r}, \quad (1) \qquad P^* = \frac{P}{P_t}, \quad (3) \qquad t^* = t\frac{Q_{l0}}{AL_r}, \quad (5) \qquad m^* = \frac{R_g T_g}{L_r P_t} m \quad (7)$$

$$s^* = \frac{s}{L_r}, \quad (2) \qquad j^* = j\frac{A}{Q_{l0}}, \quad (4) \qquad \dot{m}^* = \dot{m}\frac{R_g T_g}{Q_{l0} P_t}, \quad (6)$$

where L_r is the riser length, A is the cross-sectional area of the pipe and riser, P_t is the pressure at the top of the riser which is equal to the atmospheric pressure, s is the space parameterization along the riser length, T_g is the absolute temperature of the gas, ρ_g is the gas density, R_g is the gas constant, j stands for superficial velocity, $\dot{m}$ stands for mass flow rate, P stands for pressure, m stands for mass and t stands for time. The variables with * as a superscript are non-dimensional variables.

Pipe Governing Equations.

We first give the non-dimensional governing equation for the pipe. We consider the gas in the pipe behaving as a pressure cavity at non-dimensional pressure P_g^*, constant in position and evolving isothermically as a perfect gas. We consider a fixed control volume with the pipe and gas buffer contours as the control volume surface. For this control volume, we obtain the mass conservation equation for each of the two phases. We have to consider two different situation at the pipe. We have either continuous gas penetration from the pipe into the riser ($x^* = 0$) or partial liquid flooding of the pipe ($x^* > 0$).

Below follows the governing equations for the pipe for the conditions $x^* > 0$ and $x^* = 0$. We start with the equations for the case where $x^* > 0$. The mass conservation equation for the liquid phase is

$$-(\delta - x^*)\frac{d\alpha_p}{dt^*} + \alpha_p \frac{dx^*}{dt^*} + j_{lb}^* - 1 = 0, \tag{8}$$

where $\delta = L/L_r$. α_p is the pipe void fraction and j_{lb}^* is the non-dimensional liquid phase superficial velocity at the base of the riser. The mass conservation equation for the gas phase is

$$[(\delta - x^*)\alpha_p + \delta_b]\frac{dP_g^*}{dt^*} + P_g^*(\delta - x^*)\frac{d\alpha_p}{dt^*} - \alpha_p P_g^* \frac{dx^*}{dt^*} - \dot{m}_{g0}^* = 0, \tag{9}$$

where we used the perfect gas relation $P_g = \rho_g R_g T_g$. $\delta_b = V_b/(AL_r)$ is the non-dimensional length equivalent to the gas buffer volume V_b divided by the product of the pipe cross sectional area A by the riser length. We consider variations of pressure in the pipe only due to hydrostatic effects. Then, the momentum equation is

$$P_g^* = P_b^* + \Pi_s x^* \sin(\beta), \tag{10}$$

where P_b^* is the non-dimensional pressure at the base of the riser and the non-dimensional number Π_s is given by the equation

$$\Pi_s = \frac{\rho_l g L_r}{P_t}. \tag{11}$$

This non-dimensional number is the ratio between the hydrostatic pressure at the base of the riser when it is filled completely with liquid and the atmospheric pressure.

We can eliminate the gas non-dimensional pressure P_g^* in favor of the riser base non-dimensional pressure P_b^*, by using the equation (10). Then the liquid phase mass conservation equation is not affected, but the gas phase mass conservation equation assumes the form

$$[(\delta - x^*)\alpha_p + \delta_b]\left(\frac{dP_b^*}{dt^*} + \Pi_s \sin(\beta)\frac{dx^*}{dt^*}\right) + (P_b^* + \Pi_s x^* \sin(\beta))\left[(\delta - x^*)\frac{d\alpha_p}{dt^*} - \alpha_p \frac{dx^*}{dt^*}\right] - \dot{m}_{g0}^* = 0 \tag{12}$$

Next, we present the equations for the case $x^* = 0$. The liquid phase mass conservation equation is

$$-\delta \frac{d\alpha_p}{dt^*} + j_{lb}^* - 1 = 0. \tag{13}$$

Notice that in this case, the gas non-dimensional pressure P_g^* is equal to the non-dimensional pressure at the base of the riser. Then, we use the riser base non-dimensional pressure P_b^* instead of the gas non-dimensional pressure P_g^* in the gas phase mass conservation equation, which is

$$(\delta\alpha_p + \delta_b)\frac{dP_b^*}{dt^*} + \delta P_b^* \frac{d\alpha_p}{dt^*} + P_b^* j_{gb}^* - \dot{m}_{g0}^* = 0, \tag{14}$$

where j_{gb}^* is the gas non-dimensional superficial velocity at the base of the riser.

To close the model for the pipe, we use an implicit algebraic relation for the pipe void fraction α_p which relates it with the non-dimensional gas superficial velocity at the base of the riser j_{gb}^*, with the non-dimensional liquid superficial velocity at the base of the riser j_{lb}^* and with the non-dimensional gas pressure P_g^*, and is derived from local momentum equilibrium for each phase of a stratified flow in a pipe (Yemada and Dukler 1976, Kokal and Stanislav 1989 and others). For the case $x^* = 0$ we write

$$A_p(\alpha_p, j_{lb}^*, j_{gb}^*, P_b^*) = 0, \tag{15}$$

since in this case $P_b^* = P_g^*$. For the condition $x^* > 0$ we write the algebraic relation for α_p as

$$A_p(\alpha_p, j_{lb}^*, x^*, P_b^*, \frac{dx^*}{dt^*}) = 0. \tag{16}$$

Equations for the Riser.

For the riser, non-dimensional equations are derived from the space integration of an isothermal drift-flux model assuming quasi-equilibrium momentum balance for the two-phase flow in the riser. The mass conservation equation for the liquid phase is

$$\frac{dm_l^*}{dt^*} + \Lambda_s(j_{lu}^* - j_{lb}^*) = 0, \tag{17}$$

where j_{lu}^* is the non-dimensional liquid superficial velocity at the liquid-gas interface bounding below the gas cavity at the top of the riser when this exists, or it is the non-dimensional superficial velocity of the liquid at the top of the riser. m_l^* is the non-dimensional mass of liquid inside the riser and it is given by the equation

$$m_l^* = \int_0^{s_u^*} \Lambda_s(1-\alpha_r)ds^*, \tag{18}$$

where s^* is the non-dimensional space parameterization (vertical direction) and α_r is the void fraction along the riser. The non-dimensional number Λ_s is defined by the equation

$$\Lambda_s = \frac{\rho_l R_g T_g}{P_t}, \tag{19}$$

and is the ratio between the gas compressibility pressure and the atmospheric pressure.

The mass conservation equation for the gas phase is

$$\frac{dm_g^*}{dt^*} + P_u^* j_{gu}^* - P_b^* j_{gb}^* = 0, \tag{20}$$

where P_u^* and j_{gu}^* are, respectively, the non-dimensional pressure and the non-dimensional gas superficial velocity at the liquid-gas interface when this exists, or they are the non-dimensional pressure and the non-dimensional gas superficial velocity at the top of the riser. m_g^* is the non-dimensional mass of gas in the gas-liquid mixture filling the riser. It is given by the equation

$$m_g^* = \int_0^{s_u^*} P^*(s^*)\alpha_r ds^*, \tag{21}$$

where $P^*(s^*)$ is the non-dimensional pressure along the riser.

We assume the inertia and frictional forces small and neglect them. We consider pressure variation only due to the hydrostatic force. The linear momentum equation is

$$P_u^* - P_b^* = -\frac{\Pi_s}{\Lambda_s}[m_l^* + m_g^*]\sin\theta, \tag{22}$$

where θ is the riser inclination angle with respect to the vertical direction.

We consider the constitutive law corresponding to the drift flux model (Zuber and Findlay 1965) to relate the void fraction along the riser with the local values of the gas and liquid non-dimensional superficial velocities. At the base of the riser we have the relation

$$(1-C_d\alpha_{rb})j_{gb}^* = \alpha_{rb}[C_d(j_{lb}^*+j_{gb}^*)+U_d^*], \tag{23}$$

where α_{rb} is the void fraction at the base of the riser. At the liquid-gas interface, when it exists, or at the top of the riser we have the relation

$$(1-C_d\alpha_{ru})j_{gu}^* = \alpha_{ru}[C_d(j_{lu}^*+j_{gu}^*)+U_d^*], \tag{24}$$

where α_{ru} is the void fraction at position s_u^* (liquid-gas interface when it exists or top of the riser). For the drift flux coefficients C_d and U_d^* we use the following correlation based on experimental data (Bendiksen 1984)

$$C_d = \begin{cases} 1,05+0,15\sin\theta & \text{for } |j^*| < 3,5\frac{\sqrt{gD}A}{Q_{l0}} \\ 1,2 & \text{for } |j^*| \geq 3,5\frac{\sqrt{gD}A}{Q_{l0}} \end{cases} \tag{25}$$

$$U_d^* = \begin{cases} \frac{\sqrt{gD}A}{Q_{l0}}(0,35\sin\theta+0,54\cos\theta) & \text{for } |j^*| < 3,5\frac{\sqrt{gD}A}{Q_{l0}} \\ 0,35\frac{\sqrt{gD}A}{Q_{l0}}\sin\theta & \text{for } |j^*| \geq 3,5\frac{\sqrt{gD}A}{Q_{l0}} \end{cases} \tag{26}$$

where D is the pipe and riser diameter and $|j^*| = |j_{lb}^*+j_{gb}^*|$ at the base of the riser or $|j^*| = |j_{lu}^*+j_{gu}^*|$ at position s_u^*. Other correlations based on experimental data are given by (Chexal *et al.* 1992).

The liquid-gas interface located at s_u^* appears whenever the liquid superficial velocity becomes zero at the top of the riser. The equation for s_u^* is

$$(1-\alpha_{ru})\frac{ds_u}{dt} = j_{lu}. \tag{27}$$

Notice that this equation comes to play only in the third and fourth configurations defined previously. In order for the set of equations to match the number of dependent variables, we need two additional equations. We assume the void fraction along the riser to vary linearly with the space parameterization. Equation (18) for the non-dimensional mass of liquid in the pipe assumes the form

$$m_l^* = \frac{1}{2}\Lambda_s s_u^*[2-\alpha_{ru}-\alpha_{rb}]. \tag{28}$$

Equation (21) for the non-dimensional mass of gas in the gas-liquid mixture assume the form

$$m_g^* = \frac{1}{2}s_u^*[P_u^*\alpha_{ru}+P_b^*\alpha_{rb}], \tag{29}$$

where it is implicit the assumption of perfect gas.

Not all equations above are valid for the four configurations defined previously and illustrated in figure 1. In Tab 1 we define which equations are the governing equations for each configuration and which dependent variables are used.

Table 1 – Necessary governing equations and variables for each configuration defined in figure 1.

Configuration	Governing Equations	Dependent variables
1st	(13)-(15), (17), (20), (22)-(24), (28), (29)	$\alpha_p, j_{lb}^*, j_{gb}^*, P_b^*, \alpha_{rb}, j_{lu}^*, j_{gu}^*, m_l^*, m_g^*, s_u^* = s_t^*$, $P_u^* = P_t^*$
2nd	(8), (12), (16), (17), (20), (22)-(24), (28), (29)	$\alpha_p, x^*, j_{lb}^*, j_{gb}^* = \alpha_{rb} = 0, P_b^*, j_{lu}^*, j_{gu}^*, m_l^*, m_g^*, s_u^* = s_t^*$, $P_u^* = P_t^*$
3rd	(8), (12), (16), (17), (20), (22)-(24), (27)-(29)	$\alpha_p, x^*, j_{lb}^*, j_{gb}^* = \alpha_{rb} = 0, P_b^*, j_{lu}^*, j_{gu}^*, m_l^*, m_g^*, s_u^*$, $P_u^* = P_t^*$
4th	(13)-(15), (17), (20), (22)-(24), (28)-(29)	$\alpha_p, j_{lb}^*, j_{gb}^*, P_b^*, \alpha_{rb}, j_{lu}^*, j_{gu}^*, m_l^*, m_g^*, s_u^*$, $P_u^* = P_t^*$

Next, we have to describe when we switch from one configuration to another, or from one set of equations to another. Table 2 illustrate the conditions characterizing each configuration and the conditions to switch from the current configuration.

Table 2 – Characterization and switching conditions among configurations and correspondent set of equations.

Configuration	Characterized by	Switch from when
Frist	$x^*=0, j^*_{gb}\neq 0, \alpha_{rb}\neq 0$ and $s^*_u=s^*_t, j^*_{lu}>0$	$j^*_{gb}\to 0, \alpha_{rb}\to 0, x^*>0$ or $j^*_{lu}<0$
Second	$x^*>0, j^*_{gb}=0, \alpha_{rb}=0$ and $s^*_u=s^*_t, j^*_{lu}>0$	$j^*_{gb}>0, \alpha_{rb}>0, x^*\to 0$ or $j^*_{lu}<0$
Third	$x^*>0, j^*_{gb}=0, \alpha_{rb}=0$ and $s^*_u<s^*_t$	$j^*_{gb}>0, \alpha_{rb}>0, x^*\to 0$ or $s^*_u\to s^*_t$
Fourth	$x^*=0, j^*_{gb}>0, \alpha_{rb}>0$ and $s^*_u=s^*_t, j^*_{lu}>0$	$j^*_{gb}\to 0, \alpha_{rb}\to 0, x^*>0$ or $s^*_u\to s^*_t$

The boundary conditions are the pressure P_t at the top of the riser which is the atmospheric pressure, the gas mass flow rate $\dot{m}_{go}$ and the liquid volumetric flow rate Q_{l0} (see figure 1 for details). The boundary condition at the top of the riser in non-dimensional form is $P^*_t=1$.

ASYMPTOTIC THEORY.

For ranges of the pipe-riser system parameters there exists stable gas-liquid flows. Changes in the system parameters may lead to the lost of stability of the previously stable steady state gas-liquid flows. This lost of stability give rise to hydrodynamic instabilities characterized by limit cycles according to experiments reported in the literature. This type of behavior is typical of a supercritical Hopf bifurcation. The evolution equation associated with instabilities which arise from a supercritical Hopf bifurcation is of the Landau equation type.

In the linear stability analysis of a given steady state gas-liquid flow, we write the variables as their steady state values plus a perturbation and substitute them into the system governing equations. As a result, we obtain the perturbation governing equations. We keep only the linear terms in the governing equations for the perturbation variables and obtain a system of algebraic and differential equations. We reduce the number of variables using the algebraic equations and obtain a set of only linear differential equations. We assume the solution of the form $\mathbf{v}\exp(\lambda t)$, and the system of differential equation reduces to the eigenvalue problem

$$(\mathbf{K}+\lambda\mathbf{M})\mathbf{v}=0 \tag{30}$$

The linear stability of the steady state is dictated by the spectrum of the eigenvalue problem above. If all eigenvalues $\lambda_j=\sigma_j+i\omega_j$ have $\sigma_j<0$, the steady state is stable since the solution decays exponentially in time. In the case of the Hopf bifurcation, two complex conjugate eigenvalues cross the imaginary axis, and their real part become positive, and the perturbations of the steady state grows in time. As the perturbations grow, nonlinearity becomes important and it may bound the exponential growth of the two unstable eigenmodes, leading to a limit cycle. For system parameter values close to their critical values, the real part σ of the pair of complex conjugate eigenvalues which crossed the imaginary axis is small, and under such condition we developed an asymptotic theory which gives Landau equation

$$\frac{dA}{dt}-\sigma A(t)+\mu|A(t)|^2A(t)=0 \tag{31}$$

as the evolution equation for the amplitude $A(t)$ of the two complex conjugate unstable eigenmodes associated with the pair of complex conjugate eigenvalues with positive real part σ. The asymptotic theory furnishes the Landau equation coefficients σ and μ in terms of the steady state variables. The coefficient σ is the real and positive as implied above. For real and positive (negative) σ, the linear part of equation (31) implies an exponentially growth (decay) with time for $A(t)$ at a growth (decay) rate σ. The coefficient μ is in general complex, and if its real part is positive, the nonlinearity in equation (31) bounds the exponential growth of the linear part leading to a limit cycle, with amplitude A_c and frequency ω_c given by

$$A_c=\sqrt{\frac{\sigma}{\Re\{\mu\}}}\sim O(\sigma^{1/2}), \tag{32}$$

$$\omega_c=\omega-\frac{\Re\{\mu\}}{\Im\{\mu\}}\sigma, \tag{33}$$

where ω is the modulus of the imaginary part of the pair of complex conjugate eigenvalues with positive real part σ. $\Re\{\mu\}$ and $\Im\{\mu\}$ are, respectively, the real and imaginary parts of μ. If $\Re\{\mu\}$ is negative, the nonlinear term in equation (31) does not bound the growth of $A(t)$ for positive σ and there is no limit cycle type of solution for $A(t)$.

Since we have four different configurations (set of equations) in the model for two-phase flows in pipe-riser systems, we need to obtain four different expressions for the Landau equation coefficients σ and μ, one set for each configuration. We do this only for the first configuration, since the procedure for the other three set of equations is similar. The first step in the development of the asymptotic theory is to obtain the steady gas-liquid flow that is stable for some range of values of the system parameters.

Since we are working only with non-dimensional variables, and for simplicity, from now on we omit the superscript * from the equations.

Steady State

We consider the equations for the first configuration (see Tab. 1) to obtain the steady state. The equations which give a steady state for the first configuration are the set of equations specified in Tab. 1, but with the time derivatives set to zero. From equation (13) the steady sate superficial velocity $j_{lb,0}$ is given by

$$j_{lb,0} = 1, \tag{34}$$

and from equation (17) it follows that the steady state superficial velocity $j_{lu,0} = j_{lb,0} = 1$. From equations (14) and (18) we conclude that

$$P_{u,0}\, j_{gu,0} = P_{b,0}\, j_{gb,0} = \dot{m}_{g0}, \tag{35}$$

where $P_{u,0}$ and $P_{b,0}$ are, respectively, the steady state pressure at the top and base of the riser, and $j_{gu,0}$ and $j_{gb,0}$ are, respectively, the steady state gas superficial velocities at the top and base of the riser. The pressure $P_{u,0} = P_t = 1$, since it is a boundary condition. Then from equation (35) we can write that $j_{gu,0} = \dot{m}_{g0}$.

Since $j_{lu,0} = 1$ and $j_{gu,0} = \dot{m}_{g0}$, the drift relation at the top of the riser gives the steady state void fraction $\alpha_{ru,0}$ at the top of the riser as

$$\alpha_{ru,0} = \frac{\dot{m}_{g0}}{C_d(\dot{m}_{g0}+1)+U_{d,t}}, \tag{36}$$

where $U_{d,t}$ specifies U_d, given by equation (26), at the top of the riser.

With equations (34) and (35), the drift relation at the base of the riser (equation (23)) gives the steady state void fraction $\alpha_{rb,0}$ at the base of the riser in terms of $j_{gb,0}$ as

$$\alpha_{rb,0} = \frac{j_{gb,0}}{C_d(j_{gb,0}+1)+U_{d,b}}, \tag{37}$$

where $U_{d,b}$ specifies U_d at the base of the riser. We substitute $P_u = P_t = 1, P_b = j_{gb,0}/\dot{m}_{g0}$ and $\alpha_{rb,0}$ given by equation (37) into equations (28) and (29), which give expressions for the liquid steady state mass $m_{l,0}$ and the gas steady state mass $m_{g,0}$ in terms of $j_{gb,0}, \alpha_{ru,0}$ and $\dot{m}_{g0}$. Then, we substitute these expressions into the equation (22) to obtain a second order algebraic equation for $j_{gb,0}$. The roots are

$$j_{gb,0} = -\frac{B_1}{2B_2} \mp \frac{1}{2B_2}\sqrt{B_1^2 - 4B_0B_2}, \tag{38}$$

where we define

$$B_0 = -\dot{m}_{g0}(C_d + U_{d,b}), \tag{39}$$

$$B_1 = \sin(\theta) s_t \Pi_s \left\{ \alpha_{ru,0}\left(\frac{1}{2\Lambda_s} - \frac{1}{2}\right)(U_{d,b}+C_d) + \frac{\dot{m}_{g0}}{2\Lambda_s} + C_d + U_{d,b} \right\} - \dot{m}_{g0}C_d, \tag{40}$$

$$B_2 = \sin(\theta) s_t \Pi_s \left\{ \alpha_{ru,0}C_d\left(\frac{1}{2\Lambda_s} - \frac{1}{2}\right) + C_d - 1 \right\} + C_d. \tag{41}$$

Now, we have to decide which root to use or if each of these roots define a different steady state. Taking the limit $\dot{m}_{g0} \to 0$, we have that $B_0 \to 0$, $-B_1/(2B_2) - \sqrt{B_1^2 - 4B_0B_2}/(2B_2) \to -B_1/B_2$ ($B_1 \neq 0$ and $B_2 \neq 0$) and $-B_1/(2B_2) + \sqrt{B_1^2 - 4B_0B_2}/(2B_2) \to 0$. As the gas mass flow rate coming into the pipe goes to zero, we expect the $j_{gb,0}$ to go to zero. Therefore, the physical steady state is given by the positive sign in the equation (38) for $j_{gb,0}$. With the expression for $j_{gb,0}$ in terms of the system parameters (positive sign in equation (38)), we use equations (37), (28) and (29), respectively, to obtain $\alpha_{rb,0}, m_{l,0}$ and $m_{g,0}$ in terms of the system parameters. The steady state value for the pipe void fraction $\alpha_{p,0}$ is obtained by substituting $j_{lb,0}, j_{gb,0}$ and $P_{b,0}$ into the implicit equation (15).

The other three configurations illustrated in figure 1 do not have steady states. As we try to solve the equations (given in Tab 1, but with the time derivatives set to zero) to obtain the steady state for the other configurations, we run into inconsistencies. For example, for the fourth configuration, equations (13) and (17) give $j_{lu} = 1$, but for equation (27) to be satisfied, we need $j_{lu} = 0$, which is an inconsistency. This is not a surprise, since the second, third and fourth configurations exist only during stages of the intermittent regimes described above.

Governing Equations for the Perturbations Variables.

The steady state obtained in the previous section may be stable or not, depending on the values of the system parameters. To determine the range of the system parameters for which the steady state obtained above is stable or not, we perform a linear stability analysis. The first step would be to write the system variables as their steady state values plus a perturbation which is given as a series expansion in the parameter ε, where $\varepsilon = \sigma^{1/2}$ (order of magnitude of the limit cycle amplitude, see equation (32)) and we assume $\varepsilon << 1$.

We write a series expansion for the perturbation variables in terms of the parameter ε in the form

$$\alpha(t) = \sum_{n=1}^{\infty} \varepsilon^n \alpha_n(t,\tau), \quad (42) \qquad x(t) = \sum_{n=1}^{\infty} \varepsilon^n x_n(t,\tau), \quad (44) \qquad P_b(t) = \sum_{n=1}^{\infty} \varepsilon^n P_{b,n}(t,\tau), \quad (46)$$

$$j(t) = \sum_{n=1}^{\infty} \varepsilon^n j_n(t,\tau), \quad (43) \qquad m(t) = \sum_{n=1}^{\infty} \varepsilon^n m_n(t,\tau), \quad (45) \qquad s_u(t) = \sum_{n=1}^{\infty} \varepsilon^n s_{u,n}(t,\tau), \quad (47)$$

where $\alpha(t)$ stands for the void fractions $\alpha_p(t), \alpha_{rb}(t)$ and $\alpha_{ru}(t)$, $j(t)$ stands for the superficial velocities $j_{lb}(t), j_{lu}(t), j_{gb}(t)$ and $j_{gu}(t)$ and $m(t)$ stands for the liquid and gas masses $m_l(t)$ and $m_g(t)$. We use the two-scale method. We assume that the terms in the expansions above are functions of the time scale $\tau = \varepsilon^2 t$. Now we substitute the variables in the algebraic equations (15), (22), (23), (24), (28) and (29) by their steady state value plus their perturbation variables given by the equations (42)-(47). Then, we collect terms of the same order in ε. As a result, we obtain the system of algebraic equations

$$\mathbf{A}\mathbf{v_k} = \mathbf{f_k}, \;\; k = 1,2,3,\ldots, \tag{48}$$

where $\mathbf{v_k}^T = \{\alpha_{p,k}(t,\tau)\; j_{lb,k}(t,\tau)\; j_{gb,k}(t,\tau)\; \alpha_{rb,k}(t,\tau)\; P_{b,k}(t,\tau)\; j_{lu,k}(t,\tau)\; j_{gu,k}(t,\tau)\; \alpha_{ru,k}(t,\tau)\; m_{l,k}(t,\tau)\; m_{g,k}(t,\tau)\; s_{u,k}(t,\tau)\}$ and the non-zero elements of matrix $\mathbf{A}$ are $\mathbf{A}_{1,1} = D_1(A_p)(\alpha_{p,0}, jlb_0, jgb_0, P_{b,0})$, $\mathbf{A}_{1,2} = D_2(A_p)(\alpha_{p,0}, jlb_0, jgb_0, P_{b,0})$, $\mathbf{A}_{1,3} = D_3(A_p)(\alpha_{p,0}, jlb_0, jgb_0, P_{b,0})$, $\mathbf{A}_{1,5} = D_4(A_p)(\alpha_{p,0}, jlb_0, jgb_0, P_{b,0})$, $\mathbf{A}_{2,5} = -1$, $\mathbf{A}_{2,9} = \frac{\sin(\theta)\Pi_s}{\Lambda_s}$, $\mathbf{A}_{2,10} = \frac{\sin(\theta)\Pi_s}{\Lambda_s}$, $\mathbf{A}_{3,2} = -\alpha_{rb,0}C_d$, $\mathbf{A}_{3,3} = 1 - \alpha_{rb,0}C_d$, $\mathbf{A}_{3,4} = -C_d(j_{lb,0} + j_{gb,0}) - U_{d,b}$, $\mathbf{A}_{4,6} = -\alpha_{ru,0}C_d$, $\mathbf{A}_{4,7} = 1 - \alpha_{ru,0}C_d$, $\mathbf{A}_{4,8} = -C_d(j_{lu,0} + j_{gu,0}) - U_{d,u}$, $\mathbf{A}_{5,4} = \frac{1}{2\Lambda_s s_t}$, $\mathbf{A}_{5,8} = \frac{1}{2\Lambda_s s_t}$, $\mathbf{A}_{5,9} = 1$, $\mathbf{A}_{5,11} = \frac{1}{2}\Lambda_s[\alpha_{rb,0} + \alpha_{ru,0} - 1]$, $\mathbf{A}_{6,4} = \frac{1}{2}P_{b,0}s_t$, $\mathbf{A}_{6,5} = \frac{1}{2}\alpha_{rb,0}s_t$, $\mathbf{A}_{6,8} = -\frac{1}{2}P_{u,0}s_t$, $\mathbf{A}_{6,10} = 1$ and $\mathbf{A}_{6,11} = \frac{1}{2}[P_{b,0}\alpha_{rb,0} - P_{u,0}\alpha_{ru,0}]$. The differential operator D_j signifies the derivative with respect to the j-th argument of the function the operator is acting on. For the second and third configurations, equation (16) is an implicit differential equation, so for these configurations, matrix $\mathbf{A}$ has only five lines instead of the six lines for the first and fourth configurations. The rest of the governing equation for the perturbation variables for the first configuration are differential equations. They are

$$-\delta \frac{d\alpha_{p,k}}{dt} + j_{lb,k} = h_{1,k}, \; k = 1,2,3,\ldots, \tag{49}$$

$$-\delta - P_{b,0}\frac{d\alpha_{p,k}}{dt} + (\delta\alpha_{p,0} + \delta_b)\frac{dP_{b,k}}{dt} + j_{gb,0}P_{b,k} + P_{b,0}j_{gb,k} - = h_{2,k}, k = 1,2,3,\ldots, \tag{50}$$

$$\frac{dm_{l,k}}{dt} + \Lambda_s(j_{lu,k} - j_{lb,k}) = h_{3,k}, \; k = 1,2,3,\ldots, \tag{51}$$

$$\frac{dm_{g,k}}{dt} + P_{u,0}j_{gu,k} - j_{gb,0}P_{b,k} - P_{b,0}j_{gb,k} = h_{4,k}, \; k = 1,2,3,\ldots. \tag{52}$$

Now we solve the linear matrix equation (48) for the variables given in the vector $\mathbf{v_{k1}}^T = \{j_{lb,k}(t,\tau)\; \alpha_{rb,k}(t,\tau)\; j_{gu,k}(t,\tau)\; \alpha_{ru,k}(t,\tau)\; m_{l,k}(t,\tau)\; m_{g,k}(t,\tau)\}$ for $k = 1,2,3,\ldots$ in terms of the variables given in vector $\mathbf{v_{k2}}^T = \{\alpha_{p,k}(t,\tau)\; j_{gb,k}(t,\tau)\; P_{b,k}(t,\tau)\; j_{lu,k}(t,\tau)\}$ for $k = 1,2,3,\ldots$. We define the matrices $\mathbf{A_{11}} = [\mathbf{A}_2\; \mathbf{A}_4\; \mathbf{A}_7\; \mathbf{A}_8\; \mathbf{A}_9\; \mathbf{A}_{10}]$ and $\mathbf{A_{12}} = [\mathbf{A}_1\; \mathbf{A}_3\; \mathbf{A}_5\; \mathbf{A}_6]$, where $\mathbf{A}_k$ represents the k-th column of matrix $\mathbf{A}$. We obtain

$$\mathbf{v_{1k}} = \mathbf{B}\mathbf{v_{2k}} + \mathbf{q_k}, \;\; k = 1,2,3,\ldots, \tag{53}$$

where $\mathbf{B} = \mathbf{A_{11}}^{-1}\mathbf{A_{12}}$ and $\mathbf{q_k} = \mathbf{A_{11}}^{-1}\mathbf{f_k}$. Since for the first configuration $s_u = s_t$, the 11-th column of $\mathbf{A}$ is not used in the matrix equation (48). This equation allow us to eliminate $j_{lb,k}(t,\tau), \alpha_{rb,k}(t,\tau), j_{gu,k}(t,\tau), \alpha_{ru,k}(t,\tau), m_{l,k}(t,\tau)$ and $m_{g,k}(t,\tau)$ in favor of the variables $\alpha_{p,k}(t,\tau), j_{gb,k}(t,\tau), P_{b,k}(t,\tau)$ and $j_{lu,k}(t,\tau)$ in the other equations necessary to describe the first configuration. Next, we substitute the expression for $j_{lb,k}$ and $m_{l,k}$, given by the matrix equation (53), into the equation (51) to obtain

$$j_{lu,k} = -\frac{1}{\Lambda_s}\left\{\mathbf{B}_{5,1}\frac{d\alpha_{p,k}}{dt} + \mathbf{B}_{5,2}\frac{dj_{gb,k}}{dt} + \mathbf{B}_{5,3}\frac{dP_{b,k}}{dt}\right\} + \mathbf{B}_{1,1}\alpha_{p,k} + \mathbf{B}_{1,2}j_{gb,k} + \mathbf{B}_{1,3}P_{b,k} - \frac{1}{\Lambda_s}\left\{\frac{d}{dt}\mathbf{q}_{\mathbf{k}5} + h_{3,k}\right\} + \mathbf{q}_{\mathbf{k}1}, k = 1, \ldots \quad (54)$$

since $\mathbf{B}_{5,4} = 0$. We substitute $j_{lu,k}$, given by the equation (54), into the expression for $j_{gu,k}$ given by the matrix equation (53) ($\mathbf{B}_{2,4} \neq 0$). Then, we substitute the resulting expression for $j_{gu,k}$ together with the expression for $m_{g,k}$ (matrix equation (53)) into the differential equation (52) to obtain a differential equation only in terms of $\alpha_{p,k}, j_{gb,k}$ and $P_{b,k}$. We also substitute the expression for $j_{lb,k}$ into the differential equation (49) to obtain another differential equation only in terms of $\alpha_{p,k}, j_{gb,k}$ and $P_{b,k}$. Equation (50) is already a differential equation only in terms of $\alpha_{p,k}, j_{gb,k}$ and $P_{b,k}$. As a result, we have a system of three differential equations

$$\mathbf{M}\frac{d}{dt}\left\{\begin{array}{c}\alpha_{p,k}\\ j_{gb,k}\\ P_{b,k}\end{array}\right\} + \mathbf{K}\left\{\begin{array}{c}\alpha_{p,k}\\ j_{gb,k}\\ P_{b,k}\end{array}\right\} = \mathbf{p_k}, k = 1,2,3,\ldots, \quad (55)$$

in terms of three variables, where $\mathbf{M}_{1,1} = \mathbf{B}_{6,1} - P_{u,0}\mathbf{B}_{3,4}\mathbf{B}_{5,1}/\Lambda_s$, $\mathbf{M}_{1,1} = \mathbf{B}_{6,2} - P_{u,0}\mathbf{B}_{3,4}\mathbf{B}_{5,2}/\Lambda_s$, $\mathbf{M}_{1,3} = \mathbf{B}_{6,3} - P_{u,0}\mathbf{B}_{3,4}$ $\mathbf{B}_{5,3}/\Lambda_s$, $\mathbf{M}_{2,1} = -\delta$, $\mathbf{M}_{3,1} = -\delta P_{b,0}$, $\mathbf{M}_{3,3} = \delta\alpha_{p,0} + \delta_b$, $\mathbf{K}_{1,1} = P_{u,0}(\mathbf{B}_{3,1} + \mathbf{B}_{3,4}\mathbf{B}_{1,1})$, $\mathbf{K}_{1,2} = P_{u,0}(\mathbf{B}_{3,2} + \mathbf{B}_{3,4}\mathbf{B}_{1,2}) - P_{b,0}$, $\mathbf{K}_{1,3} = P_{u,0}(\mathbf{B}_{3,3} + \mathbf{B}_{3,4}\mathbf{B}_{1,3}) - j_{gb,0}$, $\mathbf{K}_{2,1} = \mathbf{B}_{1,1}$, $\mathbf{K}_{2,2} = \mathbf{B}_{1,2}$, $\mathbf{K}_{2,3} = \mathbf{B}_{1,3}$, $\mathbf{K}_{3,2} = P_{b,0}$ and $\mathbf{K}_{3,3} = j_{gb,0}$. The coefficients of the matrices $\mathbf{M}$ and $\mathbf{K}$ which do not appear above are zero. Once we solve the system of differential equations to obtain $\alpha_{p,k}, j_{gb,k}$ and $P_{b,k}$, we use equation (54) to obtain $j_{lu,k}$, and then the matrix equation (53) to obtain the rest of the variables of the k-th perturbation problem. Notice that the k-th perturbation problem depends on the solution of the $(k-1)$-th, $(k-2)$-th, $(k-3)$-th, $\ldots$, 1-th perturbation problems.

Landau Equation as an Evolution Equation.

To obtain the Landau equation as the evolution equation of the amplitude of the pair of unstable modes, we need to obtain the solution of the governing equations for the perturbations up to $O(\varepsilon^3)(k=3)$. For the problem of $O(\varepsilon)$, $\mathbf{f_1} = 0$, which implies that $\mathbf{q_1} = \mathbf{p_1} = 0$ and the matrix equation (55) for $k = 1$ is an homogeneous system of differential equations. We assume solution of the form $\{\alpha_{p,1}\ j_{gb,1}\ P_{b,1}\}^T = \mathbf{w_{12}}\exp(\lambda t)$, where λ and the vectors $\mathbf{w_{12}}$ and $\mathbf{w_{12}}_a^T$ are, respectively, solutions of the eigenproblems

$$(\mathbf{K} + \lambda\mathbf{M})\mathbf{w_{12}} = 0, \quad (56) \qquad \mathbf{w_{12}}_a^T(\mathbf{k}^T + \lambda\mathbf{M}^T) = 0. \quad (57)$$

For system parameters values close to the critical values, both eigenvalue problems have $\lambda_1 = \sigma + i\omega$, $\lambda_2 = \sigma - i\omega$, with $\sigma > 0$, and λ_3 real and negative. λ_1 and λ_2 correspond to the complex conjugate eigenvectors $\mathbf{z}$ and its complex conjugate $\overline{\mathbf{z}}$ of the eigenvalue problem given by equation (56). The adjoint eigenvalue problem, given by equation (57), has distinct eigenvectors, but the same eigenvalues. We assume that the eigenvalues λ_j are single roots of the characteristic equation for the eigenvalue problems (56) and (57), and under such condition,

$$\mathbf{w_{12}}_a\mathbf{M}\mathbf{w_{12}} \neq 0, \quad (58) \qquad \text{and} \qquad \mathbf{w_{12}}_a\mathbf{M}\mathbf{w_{12}} = 1 \quad (59)$$

is used as the normalization condition for $\mathbf{w_{12}}_a$. Since the term proportional to $\exp(\lambda_3 t)$ decays exponentially fast, we write the solution of equation (55), with $k = 1$, as

$$\{\alpha_{p,1}\ j_{gb,1}\ P_{b,1}\}^T = A(\tau)\mathbf{w_{12}}\exp(i\omega t) + c.c. \quad (60)$$

where $c.c.$ stands for complex conjugate and $\mathbf{w_{12}} = \mathbf{z}$. In writing the solution as above, equation (55), with $k = 1$, is satisfied with an error of $O(\varepsilon^3)$.This error has the form $\sigma\mathbf{M}\{\alpha_{p,1}\ j_{gb,1}\ P_{b,1}\}^T$ and should be added to the problem of $O(\varepsilon^3)$, as we will see below. $j_{ju,1}$ is obtained from the solution given by equation (60) with the help of equation (54) with $k = 1$. The other perturbation variables are obtained from the solution given by the equations (60) with the help of the matrix equation (53). The solution for the perturbation problem of $O(\varepsilon)$ has the form

$$\mathbf{v_1} = A(\tau)\mathbf{w_1}\exp(i\omega t) + c.c., \quad (61)$$

where the elements of the vector $\mathbf{w_1}$ were obtained as described in the above paragraph. For the problem of $O(\varepsilon^2)$, we need the non-homogeneous part of the equation (55), with $k = 2$, given by the vector $\mathbf{p_2}$, which has the form

$$\mathbf{p_2} = \mathbf{p_{20}}|A(\tau)|^2 + \mathbf{p_{22}}A(\tau)^2\exp(i2\omega t) + c.c. \quad (62)$$

with

$$\mathbf{p_{20}} = \left\{ \begin{array}{c} -P_{u,0}\mathbf{q_{11}}_3 + \bar{\mathbf{w}}_{\mathbf{1}3}\mathbf{w_{1}}_5 + \mathbf{w_{1}}_3\bar{\mathbf{w}}_{\mathbf{1}5} \\ -\mathbf{q_{11}}_1 \\ i\omega(\mathbf{w_1}_3\bar{\mathbf{w}}_{\mathbf{1}5} - \bar{\mathbf{w}}_{\mathbf{1}3}\mathbf{w_1}_5) - \bar{\mathbf{w}}_{\mathbf{1}3}\mathbf{w_1}_5 - \mathbf{w_1}_3\bar{\mathbf{w}}_{\mathbf{1}5} \end{array} \right\}, \ \mathbf{p_{22}} = \left\{ \begin{array}{c} -i2\omega\mathbf{q_{12}}_6 - P_{u,0}\mathbf{q_{12}}_3 + i2\omega\frac{P_{u,0}}{\Lambda_s}\mathbf{B}_{34}\mathbf{q_{12}}_5 + \mathbf{w_1}_3\mathbf{w_1}_5 \\ -\mathbf{q_{12}}_1 \\ i\omega\delta\mathbf{w_1}_5\mathbf{w_{v}}_1 - \mathbf{w_3}_1\mathbf{w_1}_5 \end{array} \right\} \tag{63}$$

and

$$\mathbf{q_{11}}_i = \sum_{n=1}^{6}(\mathbf{A_{12}}^{-1})_{i,n}\left[\sum_{l=1}^{10}\sum_{k=1}^{10}\mathbf{w_1}_l\mathbf{E}(\mathbf{n})_{l,k}\bar{\mathbf{w}}_{\mathbf{1}k} + \sum_{l=1}^{10}\sum_{k=1}^{10}\bar{\mathbf{w}}_{\mathbf{1}l}\mathbf{E}(\mathbf{n})_{l,k}\mathbf{w_1}_k\right] \text{ and } \mathbf{q_{12}}_i = \sum_{n=1}^{6}(\mathbf{A_{12}}^{-1})_{i,n}\left[\sum_{l=1}^{10}\sum_{k=1}^{10}\mathbf{w_1}_l\mathbf{E}(\mathbf{n})_{l,k}\mathbf{w_1}_k\right]. \tag{64}$$

We use the bilinear forms $\mathbf{w_1}^T\mathbf{E}(\mathbf{n})\mathbf{w_1}, n = 1,\ldots,6$ to make the notation more compact. The elements of these bilinear forms are $\mathbf{E}(\mathbf{1})_{l,k} = -D_{l,k}(A_p)(\alpha_{p,0}, j_{lb,0}, j_{gb,0}, P_{b,0})(l,k = 1,2,3)$, $\mathbf{E}(\mathbf{1})_{l,5} = -D_{l,4}(A_p)(\alpha_{p,0}, j_{lb,0}, j_{gb,0}, P_{b,0})(l = 1,2,3)$, $\mathbf{E}(\mathbf{1})_{5,k} = -D_{4,k}(A_p)(\alpha_{p,0}, j_{lb,0}, j_{gb,0}, P_{b,0})(k = 1,2,3)$, $\mathbf{E}(\mathbf{2})_{3,4} = \mathbf{E}(\mathbf{2})_{2,4} = C_d$, $\mathbf{E}(\mathbf{3})_{6,8} = \mathbf{E}(\mathbf{2})_{7,8} = C_d$, $\mathbf{E}(\mathbf{5})_{5,8} = -s_t/2$. The matrices $\mathbf{E}(\mathbf{4})$ and $\mathbf{E}(\mathbf{6})$ are matrices with zero elements only. The elements of the matrices $\mathbf{E}(\mathbf{n}), n = 1,\ldots,6$ not mentioned above are zero. We assume a solution of the problem of order $O(\varepsilon^2)$ in the form

$$\{\alpha_{p,2}\ j_{gb,2}\ P_{b,2}\}^T = \lambda_{\mathbf{20}}|A(\tau)|^2 + \lambda_{\mathbf{22}}A(\tau)^2\exp(i2\omega t) + c.c., \tag{65}$$

where the vectors $\lambda_{\mathbf{20}}$ and $\lambda_{\mathbf{22}}$ satisfy

$$\mathbf{K}\lambda_{\mathbf{20}} = \mathbf{p_{20}}, \tag{66}$$

$$(i2\omega\mathbf{M} + \mathbf{K})\lambda_{\mathbf{22}} = \mathbf{p_{22}}. \tag{67}$$

With the help of equation (54) with $k = 2$, we obtain $j_{lu,2}$ from $\{\alpha_{p,2}\ j_{gb,2}\ P_{b,2}\}^T$, and with the matrix equation (53) we obtain the rest of the perturbation variables of $O(\varepsilon^2)$. Therefore, the solution of the perturbation problem of $O(\varepsilon^2)$ has the form

$$\mathbf{v_2} = |A(\tau)|^2\mathbf{w_{20}} + A(\tau)^2\mathbf{w_{22}}\exp(i2\omega t) + c.c., \tag{68}$$

where $\mathbf{w_{20}}$ ($\mathbf{w_{22}}$) is obtained form $\lambda_{\mathbf{20}}$ ($\lambda_{\mathbf{22}}$) as described in the above paragraph. For the problem of order $O(\varepsilon^3)$ we need the non-homogeneous part of equation (55) with $k = 3$, given by $\mathbf{p_3}$, which has the form

$$\mathbf{p_3} = (\mathbf{M}\mathbf{w_{12}}[\sigma A(\tau) - \frac{dA}{d\tau}] + \mathbf{p_{31}}|A(\tau)|^2A(\tau))\exp(i\omega t) + \mathbf{p_{33}}A(\tau)^3\exp(i3\omega t) + c.c. \tag{69}$$

with

$$\mathbf{p_{31}} = \left\{ \begin{array}{c} \mathbf{w_1}_3\mathbf{w_{20}}_5 + \mathbf{w_{20}}_3\mathbf{w_1}_5 - i\omega\mathbf{q_{21}}_6 - P_{u,0}\mathbf{q_{21}}_3 + \frac{P_{u,0}}{\Lambda_s}\mathbf{B}_{3,4}(-i\omega\mathbf{q_{21}}_5 - \Lambda_s\mathbf{q_{21}}_1) + \bar{\mathbf{w}}_{\mathbf{1}3}\mathbf{w_{22}}_5 + \mathbf{w_{22}}_3\bar{\mathbf{w}}_{\mathbf{1}5} \\ -\mathbf{q_{21}}_1 \\ i\omega\delta(\mathbf{w_1}_5\mathbf{w_{20}}_1 + \mathbf{w_{20}}_5\mathbf{w_1}_1) - i\omega\delta(\mathbf{w_1}_1\mathbf{w_{20}}_5 + \mathbf{w_{20}}_1\mathbf{w_1}_5) - \mathbf{w_1}_3\mathbf{w_{20}}_5 - \mathbf{w_{20}}_3\mathbf{w_1}_5 + i2\omega\delta\bar{\mathbf{w}}_{\mathbf{1}5}\mathbf{w_{22}}_1 - i\omega\delta\mathbf{w_{22}}_5\bar{\mathbf{w}}_{\mathbf{1}1} \\ -i2\omega\delta\bar{\mathbf{w}}_{\mathbf{1}1}\mathbf{w_{22}}_5 + i\omega\delta\mathbf{w_{22}}_1\bar{\mathbf{w}}_{\mathbf{1}5} - \bar{\mathbf{w}}_{\mathbf{1}3}\mathbf{w_{22}}_5 - \mathbf{w_{22}}_3\bar{\mathbf{w}}_{\mathbf{1}5} \end{array} \right\}, \ \mathbf{p_{33}} = \left\{ \begin{array}{c} \mathbf{w_1}_3\mathbf{w_{22}}_5 + \mathbf{w_{22}}_3\mathbf{w_1}_5 - i3\omega\mathbf{q_{23}}_6 - P_{u,0}\mathbf{q_{23}}_3 \\ +\frac{P_{u,0}}{\Lambda_s}\mathbf{B}_{3,4}(-i3\omega\mathbf{q_{23}}_5 - \Lambda_s\mathbf{q_{23}}_1) \\ -\mathbf{q_{21}}_1 \\ i2\omega\delta(\mathbf{w_1}_5\mathbf{w_{22}}_1 + \mathbf{w_{22}}_5\mathbf{w_1}_1) - i2\omega\delta\mathbf{w_1}_1\mathbf{w_{22}}_5 \\ -i\omega\delta\mathbf{w_{22}}_1\mathbf{w_1}_5 - \mathbf{w_1}_3\mathbf{w_{22}}_5 - \mathbf{w_{22}}_3\mathbf{w_1}_5 \end{array} \right\} \tag{70}$$

and

$$\mathbf{q_{21}}_i = \sum_{n=1}^{6}(\mathbf{A_{11}}^{-1})_{i,n}\left[\sum_{l=1}^{10}\sum_{k=1}^{10}\left(\mathbf{w_{20}}_l\mathbf{E_2}(\mathbf{n})_{l,k}\mathbf{w_1}_k + \mathbf{w_{22}}_l\mathbf{E_2}(\mathbf{n})_{l,k}\bar{\mathbf{w}}_{\mathbf{1}k}\right) + \sum_{m=1}^{10}\sum_{l=1}^{10}\sum_{k=1}^{10}\mathbf{w_1}_m\left(\mathbf{w_1}_l\mathbf{E}(\mathbf{m},\mathbf{n})_{l,k}\bar{\mathbf{w}}_{\mathbf{1}k} + \bar{\mathbf{w}}_{\mathbf{1}l}\mathbf{E}(\mathbf{m},\mathbf{n})_{l,k}\mathbf{w_1}_k\right)\right], \tag{71}$$

$$\mathbf{q_{23}}_i = \sum_{n=1}^{6}(\mathbf{A_{11}}^{-1})_{i,n}\left[\sum_{l=1}^{10}\sum_{k=1}^{10}\mathbf{w_{22}}_l\mathbf{E_2}(\mathbf{n})_{l,k}\mathbf{w_1}_k + \sum_{m=1}^{10}\sum_{l=1}^{10}\sum_{k=1}^{10}\mathbf{w_1}_m\left(\mathbf{w_1}_l\mathbf{E}(\mathbf{m},\mathbf{n})_{l,k}\mathbf{w_1}_k\right)\right],$$

where $\mathbf{E_2(1)} = \mathbf{E(1)}$ is already given, $\mathbf{E_2(n)}_{l,k} = \mathbf{E(n)}_{l,k}$ for $k \geq l, n = 2, \ldots, 6,$, $\mathbf{E_2(n)}_{l,k} = \mathbf{E(n)}_{k,l}$ for $l > k, n = 2, \ldots, 6$, $\mathbf{E(m,1)}_{l,k} = D_{m,l,k}(A_p)(\alpha_{p,0}, \ldots, P_{b,0})$ $(m,l,k = 1,2,3)$, $\mathbf{E(5,1)}_{l,k} = D_{4,l,k}(A_p)(\alpha_{p,0}, \ldots, P_{b,0})$ $(l,k = 1,2,3)$, $\mathbf{E(m,1)}_{5,k} = D_{m,4,k}(A_p)(\alpha_{p,0}, \ldots, P_{b,0})$ $(m,k = 1,2,3)$, $\mathbf{E(m,1)}_{l,5} = D_{n,l,4}(A_p)(\alpha_{p,0}, \ldots, P_{b,0})$ $(m,l = 1,2,3)$ and $\mathbf{E(5,1)}_{5,5} = D_{4,4,4}(A_p)$ $(\alpha_{p,0}, \ldots, P_{b,0})$. If the coefficients of matrices $\mathbf{E(m,n)}, n = 1, \ldots, 6$ and $m = 1, \ldots, 10$ are not mentioned above, they are zero.

To avoid secular terms in the solution for $\{\alpha_{p,3}\ j_{gb,3}\ P_{b,3}\}^T$, we need to cancel out the term proportional to $\exp(i\omega t)$ in the non-homogeneous term $\mathbf{p_3}$, and in doing so we obtain

$$\mathbf{M}\mathbf{w_{12}}[\sigma A(\tau) - \frac{dA}{d\tau}] + \mathbf{p_{31}}|A(\tau)|^2 A(\tau)) = 0. \tag{72}$$

Multiplying the equation above by the adjoint vector $\mathbf{w_{12\mathit{a}}}$ and taking equation (59) into account, we obtain Landau equation with its coefficient μ given by the equation

$$\mu = -\mathbf{w_{12\mathit{a}}}\mathbf{p_{31}}. \tag{73}$$

The solution for the problem of $O(\varepsilon^3)$ has the form

$$\{\alpha_{p,3}\ j_{gb,3}\ P_{b,3}\} = \lambda_{\mathbf{33}} A(\tau)^3 \exp(i3\omega t) \tag{74}$$

and

$$(i3\omega\mathbf{M} + \mathbf{K})\lambda_{\mathbf{33}} = \mathbf{p_{33}} \tag{75}$$

is satisfied by the vector $\lambda_{\mathbf{33}}$. By using equations (53) and (54), we can obtain the rest of the perturbation variables of $O(\varepsilon^3)$ from $\alpha_{p,3}, j_{gb,3}$ and $P_{b,3}$. The asymptotic solution of $O(\varepsilon^3)$ has the form

$$\mathbf{v_3} = A(\tau)^3 \mathbf{w_{33}} \exp(i3\omega t) + c.c, \tag{76}$$

and the perturbation of the steady state with an error of $O(\varepsilon^4)$ is given as $\sum_{k=1}^{3} \mathbf{v_k}$.

STABILITY CRITERIA

For the stability of the steady state, the real part of the eigenvalues of the eigenvalue problem (56) has to be negative. The characteristic polynomial $a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ of this eigenvalue problem is a third order polynomial, what allow us to have analytical expressions for the eigenvalues. We define in terms of the coefficients $a_k, k = 0, \ldots, 3$ the quantities $R = ((a_1 a_2)/a_3 - 3a_0/a_3) - (a_2/a_3)^3/27$, $Q = (a_1/a_3)/3 - (a_2/a_3)^2/9$ and $s_1(s_2) = \{R + (-)\sqrt{Q^3 + R^2}\}^{1/3}$. If $R^2 + Q^3 \geq 0$, analytical expressions for the eigenvalues are $\lambda_1 = -(a_2/a_3)/3 + (s_1 + s_2)$ and $\lambda_2(\lambda_3) = -(a_2/a_3)/3 - (s_1+s_2)/2 + (-)i\sqrt{3}(s_1 - s_2)/2$, and if $R^2 + Q^3 < 0$, analytical expressions for the eigenvalues are $\lambda_k = 2\sqrt{-Q}\cos(\theta/3 + 2(k-1)\pi/3) - (a_2/a_3)/3, k = 1,2,3$ with $\theta = \cos^{-1}(R/\sqrt{-Q^3})$. According to the expressions above, the steady state is stable only if

$$\begin{gathered} \max\{\cos(\theta/3), \cos((\theta+2\pi)/3), \cos((\theta+4\pi)/3)\} 2\sqrt{-Q} - (a_2/a_3)/3 < 0 \qquad \text{if } R^2 + Q^3 < 0, \\ \text{and} \\ \max\{(s_1+s_2) - (a_2/a_3)/3, -(s_1+s_2)/2 - (a_2/a_3)/3\} < 0 \qquad \text{if } R^2 + Q^3 \geq 0. \end{gathered} \tag{77}$$

Lost of stability of the steady state through a Hopf bifurcation happens when a pair of complex conjugate eigenvalues have their real part to become positive as we vary the system parameters. According to the paragraph above, a Hopf bifurcation of the steady state for the present model for two-phase flows in pipe-riser systems is possible only if $R^2 + Q^3 > 0$, and in this context, we need only $-(s_1+s_2)/2 - (a_2/a_3)/3 < 0$ for the steady state to be stable.

DISCUSSION AND CONCLUSION

We were successful in obtaining Landau's equation and its coefficients as the evolution equation for the amplitude of the pair of complex conjugate unstable modes under the assumption that the steady state lose stability through a Hopf bifurcation. The lost of stability of the steady state was shown theoretically in Zakarian (2000) for a simpler two-phase flow model and is supported by the cyclic behavior of instabilities observed in experiments with gas-liquid flows in pipe-riser systems reported in the literature.

Equation (77) represents the stability criteria for the steady state of the two-phase flow model for pipe-riser systems discussed in this work, since only the first configuration has a steady state as discussed previously. We can use this stability criteria to obtain the regions in the system parameters space where the steady state is stable. For configurations of the system parameters in these regions, the system can operate without going through hydrodynamic instabilities.

Issues that remain to be solved are to verify if the eigenvalue problems related to the perturbation variables for the second, third and fourth configurations have the same structure we assumed for the eigenvalue problem for the perturbations of the first configuration, and if the real part of the complex conjugate pair of eigenvalues, if they exists, is positive and small for values of the system parameters for which the first configuration steady state has gone through a Hopf bifurcation. If these facts are not true, the asymptotic theory presented here can not be applied to the equations for the second, third and fourth configurations.

The study of dynamic behaviors more intricate than the hydrodynamic instabilities described in the introduction that the present model may have is beyond the scope of this work.

This paper is a report on work in progress.

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