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THE LAGRANGE EQUATIONS FOR SYSTEMS WITH MASS VARYING EXPLICITLY WITH POSITION: APPLICATIONS TO OFFSHORE ENGINEERING

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Abstract. *The usual Lagrange equations of motion cannot be directly applied to systems with mass varying explicitly with position. In this particular context, a naive application, without any special consideration on non-conservative generalized forces, leads to equations of motions which lack (or exceed) terms of the form* $1/2\,\dot{q}^2\,\partial m/\partial q$ *, where q is a generalized coordinate. This paper intends to discuss the issue a little further, by treating some applications in offshore engineering under the analytic mechanics point of view.*

Keywords: *Lagrange equation, variable mass with position, offshore engineering applications*

1. Introduction

Despite the well known fact that the usual Lagrange equations of motion can be directly applied to mechanical systems with mass varying explicitly with time, being invariant with respect to that sort of possibility, this is not true if the mass variation is an explicit function of position. This subtle distinction has been discussed in Pesce (2003), where the Lagrange equations of motion were obtained in an extended form[1].

Two perspectives were there followed: systems with a material type of source, attached to particles continuously gaining or loosing mass and systems for which the variation of mass is of a "nonlinear control volume type", mass trespassing a control surface. This would be the case if, for some theoretical or practical reason, a partition into sub-systems were considered. In Pesce (2003), some interesting areas of application have been cited, as those related to, tethered satellite systems and lifting-crane problems, all of them concerning the deploying or the retrieving of cables. The textile industry has also been mentioned as an important source of variable-mass systems problems in mechanics.

Two problems were there chosen to exemplify the application of the extended Lagrange equations. The first one: the deployment of a heavy cable from a reel. The second one: the impact problem of a rigid body against a liquid free surface. In this latter example, the hydrodynamic impact force may be written as a function of the added-mass of the body entering the liquid. The added mass, in this case, is an explicit function of position and the variation is related to the changing in the size (and form) of the wetted surface.

The present paper re-addresses the impact problem in the offshore engineering context, presenting some numerical simulations and assessing the discrepancies that might be produced if the system were not properly modeled. Additionally, another 'hydromechanic' problem is treated: the dynamics of a water column inside a pipe as an approximate model for the moon-pool problem.

2. The extended Lagrange Equations

For the sake of motivation, consider for the moment a very simple and hypothetical problem of a particle of mass $m(x)$, explicitly dependent on the position x, acted on by a force $F(x,\dot{x},t)$, *mass being expelled at null velocity*. The equation of motion is simply $m'(x)\dot{x}^2 + m(x)\ddot{x} = F(x,\dot{x},t)$. However, if a somewhat naive application of the usual Lagrange equation, $d\left(\partial T/\partial\dot{x}\right)/dt - \partial T/\partial x = F(x,\dot{x},t)$, were made, one would obtain $m'(x)\dot{x}^2/2 + m(x)\ddot{x} = F(x,\dot{x},t)$, in an obvious disagreement with respect to the first and correct equation of motion derived from Newton's Law. The

[1] This extended form also comprises the (hypothetical) case of an explicit variation with respect to velocity.

reason for such a somewhat unexpected discrepancy could be easily guessed: the usual form of Lagrange Equation is not the most general form that could be conceived, concerning a system presenting variation of mass, explicitly dependent on position. In this simple example of a one-degree-of-freedom system, we could infer that the correct 'Lagrange' equation should be written $d\left(\partial T/\partial \dot{x}\right)/dt - \partial T/\partial x = F(x,\dot{x},t) - m'(x)\dot{x}^2/2$.

The extended Lagrange equations of motion can then be derived in a general case of a system of particles, for which mass is explicitly dependent on position (as well as on velocity), $m_i = m_i(q_j;\dot{q}_j;t)$. Consider a system of N particles of mass m_i. Let P_i be the corresponding position in a given inertial frame of reference and $\mathbf{p}_i = m_i\mathbf{v}_i$ the momentum. By extending Levi-Civita's form of Newton's law to cases when mass is gained or lost with no null velocity, the principle of virtual work applied to D'Alembert's Principle can be written

$$\sum_i \left(\frac{d\mathbf{p}_i}{dt} - \mathbf{F}_i \right) \cdot \delta P_i = \mathbf{0} \tag{1}$$

where $\mathbf{F}_i = \mathbf{f}_i + \mathbf{h}_i$, being $\mathbf{f}_i$ the sum of all active forces acting on P_i, and $\mathbf{h}_i = \dot{m}_i\mathbf{v}_{oi}$, a reactive force, proportional to the rate of variation of mass with respect to time and to the velocity $\mathbf{v}_{oi}$ of the expelled (or gained) mass. Note that the reactive force known as Metchersky's force, in the Russian technical literature, is usually written as function of relative velocities, in the form $\Phi_i = \dot{m}_i\left(\mathbf{v}_{oi} - \mathbf{v}_i\right) = \mathbf{h}_i - \dot{m}_i\mathbf{v}_i$. Under this latter interpretation, Eq. (1), would be written, Cveticanin (1993)),

$$\sum_i \left(m_i \frac{d\mathbf{v}_i}{dt} - (\mathbf{f}_i + \Phi_i) \right) \cdot \delta P_i = \mathbf{0} \tag{2}$$

The extended Lagrange equation may be derived as, Pesce (2003),

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = \hat{Q}_j \, ; \qquad j = 1,\ldots,M \\ &\hat{Q}_j = \sum_i \left(\mathbf{f}_i + \dot{m}_i\mathbf{v}_{oi}\right)\cdot\frac{\partial P_i}{\partial q_j} + \sum_i \left\{ \frac{1}{2}\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_j}(\mathbf{v}_i)^2\right) - \frac{1}{2}\frac{\partial m_i}{\partial q_j}(\mathbf{v}_i)^2 \right\} \end{aligned} \tag{3}$$

being $\mathbf{v}_i = \mathbf{v}_i(q_j;\dot{q}_j;t); j = 1,\ldots,M$, where q_j denotes a generalized coordinate and $\hat{Q}_j$, the respective non-conservative generalized force. This generalized force includes all active forces $\mathbf{f}_i$ and reactive forces $\dot{m}_i\mathbf{v}_{oi}$, due to addition or expelling of mass, with 'absolute' velocity $\mathbf{v}_{oi}$. These equations recover the derivation provided in Cveticanin (1993), that is valid for the simpler case of mass only explicitly dependent on position, not on velocity.

3. Examples of applications in offshore engineering

We proceed exemplifying two cases in offshore engineering where the present analysis might be relevant. The first case is a first approximation to *the moon-pool problem*. The second one is an important problem in hydromechanics, *the impact of a rigid body against the water free surface*.

3.1 The dynamics of the water column inside moon-pools and free-surface piercing pipes

Moon-pools are commonly found in many floating offshore structures as in pipe-laying and work barges. Figure 1 presents a mono-column oil production platform, with a cylindrical moon-pool. Pipes and cables are suspended through the moon-pool to the sea bottom. The main purpose is to provide safer operational conditions, regarding the action of waves. Nevertheless, the water column inside the moon-pool may resonate due to the wave action and to the motions of the floating platform. Resonance in this case should be avoided. Another interesting related problem is the dynamics of free surface piercing pipes used as elements of hydro-electrical power devices driven by the action of waves; see e.g., Tannuri and Pesce (1995). In this latter case, however, resonance is the key to a good performance. Either case, the nonlinear dynamics of the water column must be modeled properly. For the purpose of the present paper, we shall consider the simplest case of a free-surface piercing pipes opened to the atmosphere. Only the unforced problem will be addressed. The forced problem, due to the action of ocean waves, might then be readily assessed.

Consider an open vertical circular pipe of internal radius R piercing a quiescent external free surface of an incompressible, inviscid liquid; Fig. 2. Let H be the draft of the pipe. Let g be the acceleration of gravity. For simplicity, let $\zeta(t)$ describe the position of the free surface of the column of liquid in the interior of the pipe. Clearly, a simplified model with just one degree of freedom (one generalized coordinate) can be used, $\zeta(t)$. Other free surface vibration modes are not considered in this simplified model.

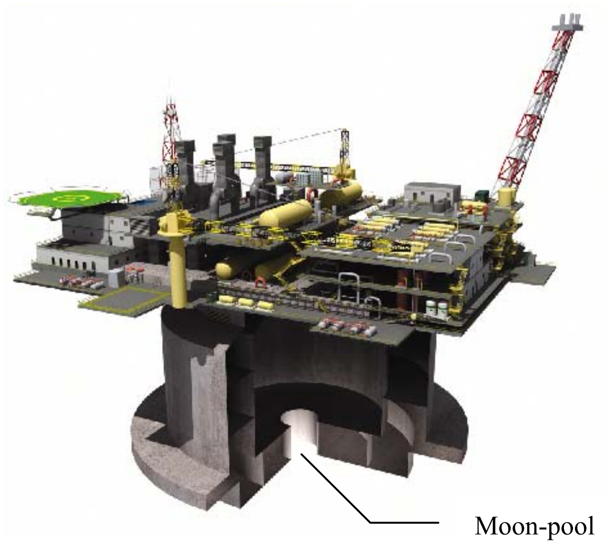

Figure 1. A mono-column, floating oil production platform. The risers and umbilical cables that connect the production plant to the well heads are suspended from the platform through the moon-pool.

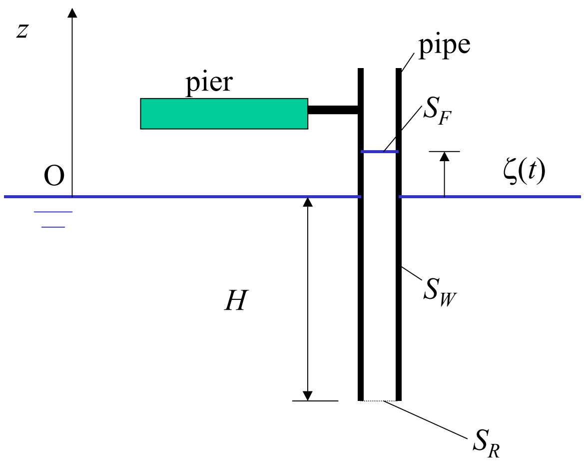

Figure 2. The free surface piercing, open pipe problem.

Before the Lagrangean approach is applied, the equation of motion is derived from the point of view of potential theory in hydrodynamics. This equation will serve as a basis of comparison.

3.1.1 The hydrodynamic approach

Take the material sub-system as composed solely by the liquid inside the pipe. That is, the liquid that in a given instant fills the volume Ω bounded by $\partial\Omega = S = S_F \cup S_R \cup S_W$. S_F is the (material and non-permeable) free surface, $z = \zeta(t)$. S_W is the material, fixed and non-permeable surface, corresponding to the interior wetted surface of the pipe and S_R the non-material (permeable) fixed control surface at the lower end of the pipe, given by $z_R = -H$. An exchanging flux of mass clearly exists between the sub-system and the external fluid. Note that the vertical components of the outwardly positive normal unit vector are $n_z = 1$ on S_F and $n_z = -1$ on S_R. Let the flow be irrotational and $\phi(z)$ the potential velocity function. The kinematic (Neuman) boundary condition on S_F is $\phi_z = \zeta_t = \dot{\zeta}$. The velocity potential, inside the pipe, can then be written $\phi(x, y, z, t) = z\dot{\zeta}$. Notice that $\phi_t = z\ddot{\zeta}$.

Let the fluid be unbounded in the far field. The dynamic pressure on S_R is given by $p_D(x, y)\big|_{S_R} = -\frac{1}{2}\rho\dot{\zeta}^2$. Pressure on S_F is taken as null, as usual. Therefore, from momentum considerations, the dynamic equation for $\zeta(t)$ is readily derived. In fact, let Q_z be the linear momentum of the fluid inside the pipe. Then, from classical potential hydrodynamics, see, e.g. Newman (1974),

$$\frac{dQ_z}{dt} = \rho \frac{d}{dt} \int_S \phi n_z dS = W + F_R - q\,, \tag{4}$$

where

$$\begin{aligned} F_S &= -\rho g A \zeta \\ F_D &= -\int_{S_R} p_D(x,y,z,t) n_z dS = -\rho A \frac{1}{2} \dot{\zeta}^2 \end{aligned} \tag{5}$$

are respectively the forces due to the differential hydrostatic pressure and to the hydrodynamic pressure applied to the water column, on S_R, and

$$q = \rho \int_{S_R} \phi_z (\frac{\partial \phi}{\partial n} - U_n) dS = -\rho \int_{S_R} \phi_z{}^2 dS = -\rho A \dot{\zeta}^2 \tag{6}$$

is the flux of linear momentum across the fluid boundary, S_R. The mass of fluid inside the pipe at a given instant is an explicit function of position, $M = \rho A(\zeta + H)$. Therefore, the time rate of linear momentum inside the pipe can be directly calculated,

$$\frac{dQ_z}{dt} = \frac{d}{dt}\left(\rho A(\zeta + H)\dot{\zeta}\right) = \rho A(\zeta + H)\ddot{\zeta} + \rho A \dot{\zeta}^2 . \tag{7}$$

Note that this result could also be achieved from hydrodynamic theory, by recalling that the derivative and integral signs are interchangeable on the fixed control surface S_R. Therefore,

$$\begin{aligned} \frac{dQ_z}{dt} &= \rho \frac{d}{dt} \int_S \phi n_z dS = \rho \frac{d}{dt} \int_{S_F} \phi n_z dS + \rho \frac{d}{dt} \int_{S_R} \phi n_z dS = \\ &= \rho \frac{d}{dt} \int_{S_F} \phi dS - \rho \frac{d}{dt} \int_{S_R} \phi dS = \rho \frac{d}{dt} \int_{S_F} \phi dS - \rho \int_{S_R} \frac{\partial \phi}{\partial t} dS = \\ &= \rho \frac{d}{dt} \int_{S_F} \zeta \dot{\zeta} dS + \rho \int_{S_R} H \ddot{\zeta} dS = \rho A\left(\zeta \ddot{\zeta} + \dot{\zeta}^2 + H \ddot{\zeta}\right) = \rho A(\zeta + H)\ddot{\zeta} + \rho A \dot{\zeta}^2 \end{aligned} \tag{8}$$

Collecting terms from Eqs. (4)-(7), we obtain

$$\rho A(\zeta + H)\ddot{\zeta} + \rho A \dot{\zeta}^2 = -\rho A g \zeta - \frac{1}{2} \rho A \dot{\zeta}^2 + \rho A \dot{\zeta}^2 , \tag{9}$$

This reduces to the following nonlinear homogeneous equation

$$\ddot{\zeta} + \frac{1}{2} \frac{\dot{\zeta}^2}{(\zeta + H)} + g \frac{\zeta}{(\zeta + H)} = 0 . \tag{10}$$

Let $\eta(t) = \zeta(t)/H$ be the nondimensional free surface position. Defining the nondimensional time as $\tau = \omega t$, $\omega = \sqrt{g/H}$, Eq. (10) may be written in nondimensional form as,

$$\ddot{\eta} + \frac{1}{2} \frac{\dot{\eta}^2}{(\eta + 1)} + \frac{\eta}{(\eta + 1)} = 0 . \tag{11}$$

The constant ω can be readily recognized as the dimensional natural frequency of the corresponding linear oscillator $\ddot{\eta} + \eta = 0$, obtained from Eq. (11) in the case of small displacements and small velocities. Note also that the term that is quadratic in velocity is, in fact, conservative. This could be easily proved. Equation (11) is valid for $-1 < \eta$.

A singular behavior, leading to infinity acceleration, arises when $\eta = -1$, i.e. $\zeta = -H$. Physically, this corresponds to the water-column surface reaching the bottom of the pipe, the mass of the system becoming zero. Beyond this point, a cavity would form, and a proper modeling should consider this other highly nonlinear phenomenon.

3.1.2 The Lagrange equation approach

From another point of view, the whole fluid dynamics may be modeled as a single degree of freedom (hydro-) mechanical system, such that $T = \frac{1}{2}\rho A(\zeta + H)\dot{\zeta}^2$ is the kinetic energy inside the pipe. One obtains

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{\zeta}} = \rho A(\zeta + H)\ddot{\zeta} + \rho A\dot{\zeta}^2 \\ &\frac{\partial T}{\partial \zeta} = \frac{1}{2}\rho A\dot{\zeta}^2 \end{aligned} \quad . \tag{12}$$

Note that, if the system were defined starting from the kinetic energy, the mass dependence on position could not be promptly recognized. As can be clearly seen, the quantity

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\zeta}} - \frac{\partial T}{\partial \zeta} = \rho A\left((\zeta + H)\ddot{\zeta} + \frac{1}{2}\rho A\dot{\zeta}^2\right), \tag{13}$$

that arises when the usual Euler-Lagrange equation is applied, is not the time rate of change of linear momentum inside the pipe, which is given by Eq. (13). To this quantity it should be added $\frac{1}{2}\rho A\dot{\zeta}^2$, that is exactly the quantity one would obtain from the additional term $\frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\mathbf{v}_i^2$, that appears on the right hand side of equation (3a). In fact,

$$\frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\mathbf{v}_i^2 = \frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\dot{\zeta}^2 = \frac{1}{2}\frac{\partial}{\partial \zeta}\left(\sum_i m_i\right)\dot{\zeta}^2 = \frac{1}{2}\rho A\frac{\partial}{\partial \zeta}\left(\int_{-H}^{\zeta} dz\right)\dot{\zeta}^2 = \frac{1}{2}\rho A\frac{\partial}{\partial \zeta}(\zeta + H)\dot{\zeta}^2 = \frac{1}{2}\rho A\dot{\zeta}^2 \tag{14}$$

To consistently apply the extended Lagrange Equation (3a), we must consider the equivalent non-conservative generalized force, according to Eq. (3b), that in this case reads,

$$\begin{aligned} \hat{F}_z &= f + \dot{m}v_o - \frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\mathbf{v}_i^2 = (F_S + F_D) + \dot{m}v_o - \frac{1}{2}\frac{\partial m}{\partial \zeta}\dot{\zeta}^2 = \\ &= \left(-\rho Ag\zeta - \frac{1}{2}\rho A\dot{\zeta}^2\right) + \left(\rho A\dot{\zeta}^2\right) - \left(\frac{1}{2}\rho A\dot{\zeta}^2\right) = -\rho Ag\zeta \end{aligned} , \tag{15}$$

Note that, in this case, the term $\frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\mathbf{v}_i^2$ is, quantitatively, half the momentum flux and exactly the same as that corresponding to the dynamic pressure. Note also that only the (conservative) hydrostatic term is left.

Collecting results, from Eqs. (13) and (15), Eq. (3a) *recovers the consistent dynamic equation*, given by Eqs. (10) or (11). Otherwise, *disregarding the term* $\frac{1}{2}\sum_i \frac{\partial m_i}{\partial \zeta}\mathbf{v}_i^2$ *would lead to the erroneous equation of motion*,

$$\ddot{\eta} + \frac{\dot{\eta}^2}{(\eta + 1)} + \frac{\eta}{(\eta + 1)} = 0 . \tag{16}$$

Apart the conceptual correctness, from the point of view of practical application, significant differences between Eq. (10) and Eq. (16) arise only if the velocity is large enough.

Despite the fact results could be consistently recovered, a rigorous generalization of Eq. (3) to continuum systems is not straightforward as it could appear through this simple example. A rigorous treatment of Hamilton Principles in Continuum Mechanics can be found in Seliger and Whitham (1967). However, to the present date, and to the author's knowledge, no theoretical extension has been made considering the case of continuum systems with variable mass as an explicit function of coordinates and velocities.

Figure 3 presents a comparison between results obtained by using Eq. (10) and Eq. (16). The phase trajectories are closed curves, since no dissipation was considered. The quadratic terms in velocity are conservative, as already anticipated. For all initial displacements, the acceleration attains a maximum when the water column level reaches its minimum value (mass inside the pipe is minimum), as already mentioned. This is exactly what is observed in reality.

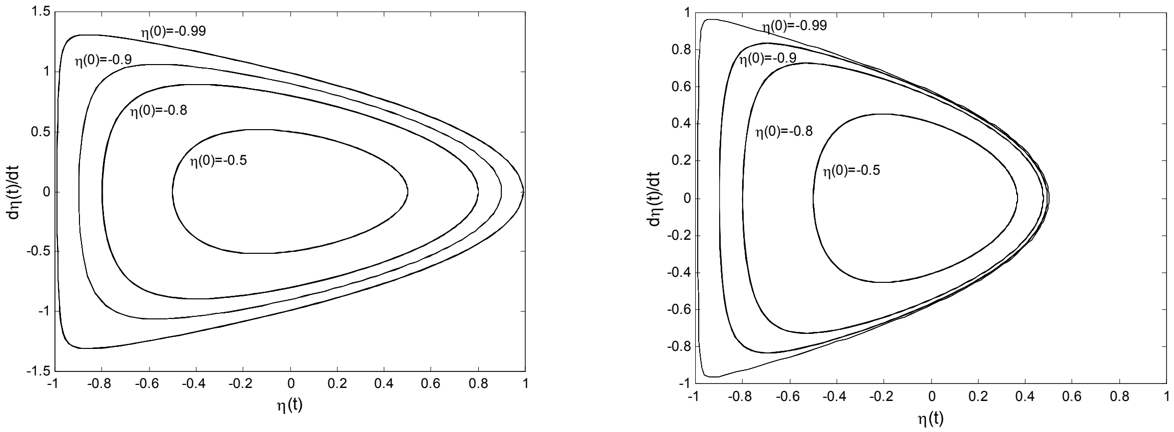

Figure 3. Phase portraits of the water column dynamics. Comparison between results from the consistent (left) and the erroneous equations (right). Initial conditions: $\eta(0) = -0.99....-0.5; \dot{\eta}(0) = 0$.

3.3 The impact of a rigid body against the water surface

Consider a body impacting a quiescent free surface of a liquid. In the offshore engineering context, important examples that could be mentioned are the deployment of lifeboats from platforms, ship slamming and wave impacts against structures. Von Karman (1929) first addressed the simplest problem (of an impacting rigid body), in order to estimate the loading on seaplane floaters during “landing”.

The duration of the impact is so short that inertia forces dominate viscous ones. This makes consistent to treat the problem within potential flow theory. As well known, it is usual practice to treat potential hydrodynamic problems involving motion of solid bodies within the frame of system dynamics. This is done whenever a finite number of generalized coordinates can be used as a proper representation for the motion of the whole fluid. Terming this approach as 'hydromechanical' the impact force acting upon the body, for a purely vertical impact, may be written $F_z = -d(M_{zz}W)/dt$ (see, e.g., Faltinsen (1990), chapter 9), being W the (positively) downward vertical velocity and M_{zz} the corresponding added mass. Note that in this case the added mass may be written as an explicit function of the position of the body and has to be determined at each instant of time, during the impact phenomenon. This is not an easy task, as the hydrodynamic problem is geometrically nonlinear due to the presence of the free surface and the moving body.

A usual approximate approach is due to Wagner (1931). Under Wagner’s approach the impact is modeled as a mathematical impulse idealization, enabling a time jump in velocity potential to occur. The impacting surface of the body is taken as the equivalent surface of a ‘time-varying floating plate’. In other words, the interaction problem is treated as the ‘impact of a floating plate’ whose area changes in time. The usual free-surface condition is replaced by an equipotential boundary condition, say $\phi = 0$, corresponding to the limit of infinity frequency in the sense of the wave radiation problem. At the very start stage, the condition $\phi_t = 0$ on an equipotential control surface that replaces the actual free surface is valid, except close to the body intersection, where jets are formed. If this exception is not taken into account, the impact force is derived as $F_{0z} = -1/2 \times W\dot{M}_{zz} - M_{zz}\dot{W}$, an erroneous result. Actually, to impose $\phi_t = 0$ everywhere, even at the body intersection, is equivalent to disregard the flux of kinetic energy through the jets. A more detailed analysis is presented in Pesce (2004). A thorough mathematical review may be found in Korobkin and Pukhnachov (1988).

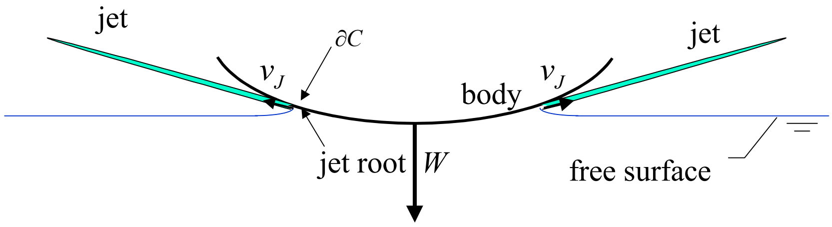

Figure 4. A rigid body impacting a quiescent free surface of a liquid.

The impact problem can also be formulated under the Lagrangian formalism, recalling the explicit added mass dependence on the position of the body. However, the same erroneous result would be obtained if Lagrange equation were not properly applied, namely, the extended form given by Eq. (3); Pesce, (2003).

Taking, for simplicity, the purely vertical impact case of an axi-symmetric rigid body against a free surface, let ζ be defined as the (positive downward) vertical displacement of the body into the water, measured from the quiescent free surface. Let $W(t)$ be the downward vertical velocity. The kinetic energy in the bulk of the liquid may be written as

$$\begin{aligned} T &= \frac{1}{2} M_{zz} W^2 \\ M_{zz} &= M_{zz}(\zeta) \,. \\ \zeta &= \int_{0^+}^{t} W dt \end{aligned} \tag{17}$$

The added mass is consistently defined *in the bulk of the liquid*, at each instant of time, by taking into account the so-called wetted correction, due to the marching of the jet root. As observed, the correct Lagrange equation approach is to use Eq. (3), such that the total vertical force applied by the body (and the jets) on the bulk of the fluid is given by

$$-F_z^{\ B} = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta} W^2 - 2\dot{m} v_J \sin\alpha \,. \tag{18}$$

The fourth term[2] corresponds to the reactive force applied by the jets; $\dot{m}$ is the flux of mass through the jets and v_J the absolute velocity of the fluid particles at the jet root; α is the instantaneous angle of the jets with respect to the horizontal.

The force applied by the bulk of fluid on the body is then, simply,

$$F_z = -\frac{d}{dt}\left(\frac{\partial T}{\partial W}\right) + \frac{\partial T}{\partial \zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta} W^2 \,. \tag{19}$$

Equation (19) transforms, as expected, into

$$F_z = -\frac{d}{dt}\left(M_{zz} W\right) + \frac{1}{2} W^2 \frac{dM_{zz}}{d\zeta} - \frac{1}{2}\frac{dM_{zz}}{d\zeta} W^2 = -\frac{d}{dt}\left(M_{zz} W\right) \tag{20}$$

The third term appearing on the right hand side of Eq. (20), if not considered, would lead to an *erroneous* assertive, according to which,

$$F_z = -\frac{1}{2}\frac{dM_{zz}}{dt} W - M_{zz}\frac{dW}{dt} \,. \tag{21}$$

As mentioned, Eq. (19) recovers the expected result. Note that in the present case the changing in the added mass is due to an actual changing of size and shape of the body in contact with the liquid. This should not be confused with common cases where the body has the size and shape invariant and the added mass varies according to its proximity to material surfaces; see, e.g., the analysis in Lamb (1932), art. 137. The computation of the function $M_{zz}(\zeta)$ is not an easy task, as the wetted surface of the body is not known a priori.

Under Wagner's approximation, the equivalent floating plate of varying size has to be determined. For bodies of regular shape, as edges, cylinders, spheres, asymptotics and singular perturbation methods can be applied successfully; see, e.g., see Faltinsen and Zhao (1997) or Pesce et al (2003), for a brief review. For generic geometric forms however, numerical schemes have to be used to solve the nonlinear hydrodynamic problem.

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[2] This term is, in fact, small. For the particular and important case of a circular cylinder of radius *R*, e.g., it can be proved, from the asymptotic analysis by Molin et al. (1996), that the vertical force, per unit length, applied by the jets on the bulk of fluid is of order $O\left(\varepsilon\pi\rho R W^2 \sin\alpha\right)$, where $\varepsilon = \sqrt{Wt/R}$ is a small parameter measuring a short scale of time. *Contrarily*, the energy flux is of order $G = O\left(\pi\rho R W^3\right)$ and $\frac{d}{dt}\left(M_{zz} W\right) = O\left(\varepsilon^{-2}\pi\rho R W^2\right)$.

As a simple example, we take the case of a sphere of radius R and mass m, reaching the free surface with initial velocity W_0. Let the nondimensional time be defined as $\mathrm{t} = W_0 t/R$, such that the nondimensional position, velocity and acceleration are given by $\eta = \zeta/R$, $\dot{\eta} = \frac{d\eta}{d\mathrm{t}} = \frac{1}{W_0}\frac{d\zeta}{dt}$ and $\ddot{\eta} = \frac{d^2\eta}{d\mathrm{t}^2} = \frac{R}{W_0^2}\frac{d^2\zeta}{dt^2}$.

Asymptotics techniques and similarity theory, applied to the impacting sphere problem to calculate the added mass function under Wagner's approach, together with the generally valid Eq. (28), leads to the following consistent nondimensional equation of motion, (Casetta (2004)),

$$\ddot{\eta} + \frac{\frac{9\sqrt{3}}{2\pi}\eta^{1/2}\dot{\eta}^2}{\beta + \frac{3\sqrt{3}}{\pi}\eta^{3/2}} = 0 \qquad (22)$$

where $\beta = m/m_D$ is the nondimensional mass ratio coefficient, with $m_D = \rho\frac{4}{3}\pi R^3$ as the displaced mass of a totally immersed sphere. This is the only parameter in Eq. (22).

However, if the added mass varying term were not properly considered, such that Eq. (21) would supposed to hold, the equation of motion would read,

$$\ddot{\eta} + \frac{1}{2}\frac{\frac{9\sqrt{3}}{2\pi}\eta^{1/2}\dot{\eta}^2}{\beta + \frac{3\sqrt{3}}{\pi}\eta^{3/2}} = 0 . \qquad (23)$$

The dimensional impacting force is then given by $F_z(t) = \frac{mW_0^2}{R}\frac{d^2\eta}{d\mathrm{t}^2}$. Or else, if written in terms of the body weight, its is given by $F_z(t) = \left(\mathbf{F_r}^2\frac{d^2\eta}{d\mathrm{t}^2}\right)mg$, where $\mathbf{F_r} = W_0/\sqrt{gR}$ is the Froude number. Note that Eq. (22) was asymptotically derived assuming small submergence, say $\eta < 0.2$. In this stage the impacting force reaches its maximum value. Moreover, the impacting force usually dominates the buoyancy force and that is the reason why buoyancy has not been considered.

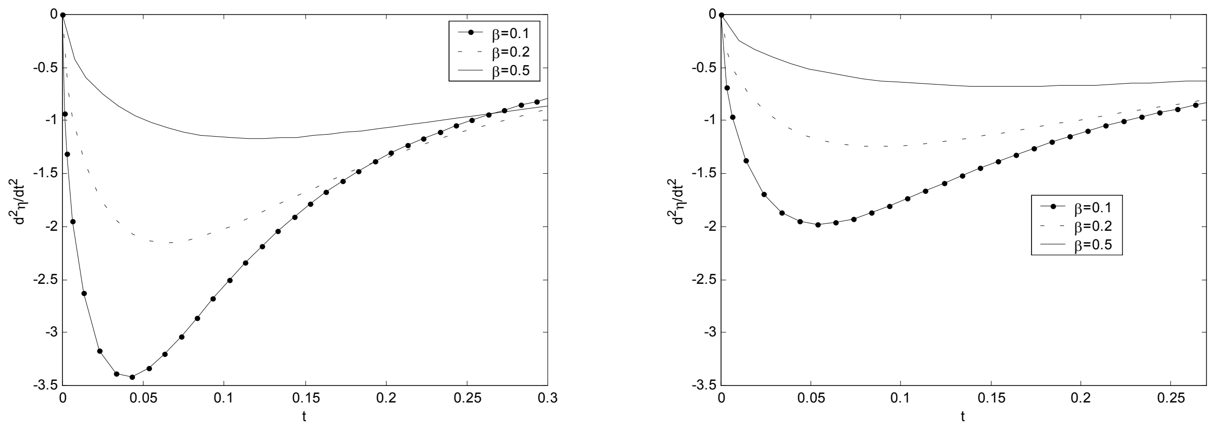

Figure 5. Nondimensional acceleration, $\ddot{\eta} = d^2\eta/d\mathrm{t}^2 = \left(R/W_0^2\right)d^2\zeta/dt^2$, of an impacting sphere of radius R striking the water surface at velocity W_0, as function of nondimensional time, $\mathrm{t} = W_0 t/R$. Comparison between results obtained with the consistent (left) and the erroneous equations (right). $\beta = 3m/\left(4\rho\pi R^3\right)$.

As can be easily inspected from Eq. (22), the impacting force peak decreases with the mass ratio and increases with the square of the Froude number. In fact, from Eq. (22), the impacting force peak is of order $F_I = O(\beta^{-1}\mathbf{F_r}^2 mg) = O(\mathbf{F_r}^2 m_D g)$. On the other hand, the maximum buoyancy force (totally immersed sphere) is given by $F_B = m_D g = \beta^{-1}mg$. Therefore, $F_I/F_B \approx O(\mathbf{F_r}^2) >> 1$, for high-speed impacts. As a figure, if the sphere is dropped (in vacuum) from a height H, we obtain $\mathbf{F_r}^2 = 2H/R$. Equation (22) is to be integrated under initial conditions $\eta(0) = 0$ and $\dot{\eta}(0) = 1$. Figure 5 exemplifies the large discrepancies, existing between the results obtained from both equations: the consistent equation, Eq. (22) and the erroneous one, Eq. (23). Note also that, for usual offshore and naval engineering applications, practical relevance exists for mass ratio values smaller than 1.

4. Conclusions

Through simple modeling of typical problems in offshore engineering, this work exemplified how a non-proper use of the Lagrangean formalism may lead to important discrepancies in formulating the equations of motions. This would be always the case whenever we treat mechanical systems with mass explicitly dependent on position. Despite such a strong assertive, the corresponding extended form of the Lagrange equation is not well known, being absent in almost all textbooks in classical mechanics.

5. Acknowledgements

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