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ABSORBER THEORY APPLIED TO BEAM VIBRATION

Driemeier, Larissa
driemeie@usp.br

Alves, Marcilio
maralves@usp.br

Dias, Carlos Alberto Nunes
candias@usp.br

Grupo de Mecânica dos Sólidos e Impacto em Estruturas
Escola Politécnica da USP Departamento de Mecatrônica e Sistemas Mecânicos
Av Prof Mello Moraes 2231 Cidade Universitária Butantã São Paulo SP Brasil

Abstract. *The design of vibration absorbers has been recurrently received attention from researchers due to the importance of efficiently control undesired vibration in structures. In this paper, the well-known two-degree of freedom vibration absorber theory is extended to multi-degree of freedoms and applied to the supression of beam vibrations. The analytical solution is based on the Euler/Bernoulli Beam Theory. A finite element model of the problem also presented here enriches the theoretical solution by including additional effects due to beam rotational inertia and shear deformation, approaching to the Timoshenko Beam Theory. A small scale experiment, with an unbalanced electrical fun as the excitation source of a beam, is explored in order to validate the presented theory. Finally, a parametric study is conducted in order to assist the design of vibration absorbers.*

***Keywords**: Absorber, Beam Theory, FEM Model, Experimental Results*

1. Introduction

The work of Den Hartog in 1956 made well known the two-degree-of-freedom dynamic absorber and it has been a source of inspiration for many theoretical developments and practical applications in the field of vibration suppression. Jalili and Olgac (1999) applied the theory in a wider context by introducing more than one absorber but still based on multi-degree systems. On the other hand, Malfa *et al* (2000) solved a practical case of vibration suppression using a continuous system as a single absorber in a circuit board. However, the authors did not model their solution as a continuum system despite asserting that the observed good results were promising and could be extended for bigger structures as used in naval or civil engineering systems. Recently, Ren (2001) introduced an additional external damper on an ordinary absorber and proved that the existence of two no-damping-dependent points in the amplitude response curves is maintained.

The present paper introduces another kind of variance of the classical absorber of Den Hartog by extending the ordinary two-degree of freedom absorber concept to a continuous beam, so that the theory can be verified in a more practical situation of distributed-parameter systems. This generalization leads to a boundary value mathematical problem, instead of the more easily handled matrix equilibrium equations of lumped systems. It is presented in detail the alluded mathematical model and its solution. A practical case of vibration suppression using a continuous system is solved in order to show the application of the solution here developed.

2. Vibrating Beam Theory

Considering the case of a Euler/Bernoulli slender beam (with no rotational inertia and no shear deformation effects) in bending vibration, the dynamic equilibrium differential equation can be expressed by (Clough, 1975):

$$m.\ddot{v}(x,t) + c.\dot{v}(x,t) + EI.v''''(x,t) = p(x,t) \tag{1}$$

where m is the mass per unit length [Kg/m=N.s^2/m^2] , c is the damping per unit length [N.s/m^2], EI is the bending stiffness [N.m^2], $v(x,t)$ is the beam transversal displacement [m] and $p(x,t)$ is the distributed load over the beam span [N/m].

In order to apply modal superposition to solve this equation, it is necessary to obtain natural frequencies and shape modes for the undamped free vibration case, such that Eq. (1) becomes:

$$m.\ddot{v}(x,t) + EI.v''''(x,t) = 0 \tag{2}$$

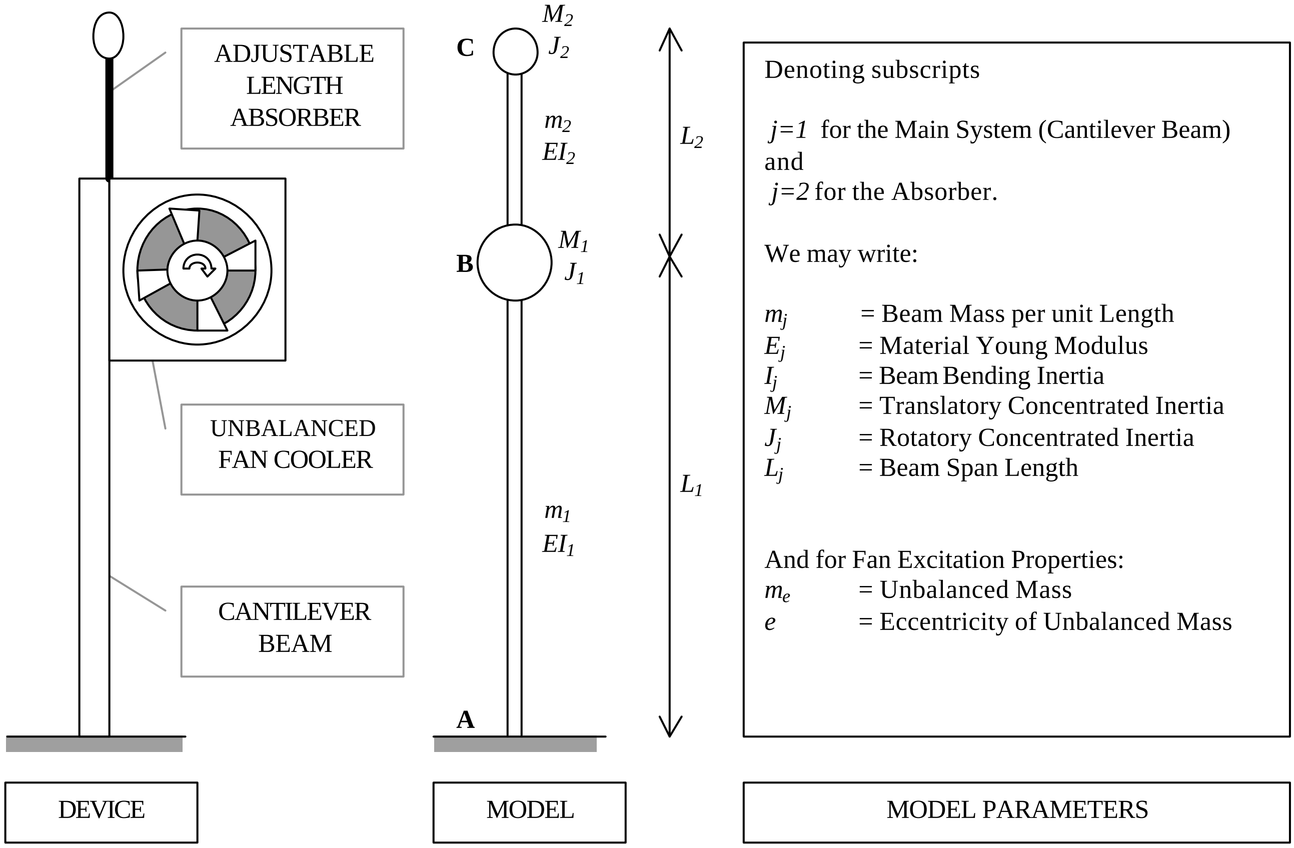

Figure 1 The Experimental Device and corresponding simplified Theoretical Model

The solution of Eq.(2) for a single span beam is harmonic and given by the well known separation-of-variables solution $v(x,t)=f(x).y(t)=f(x).\text{sen}(w.t)$, such that,

$$v(x,t)=f(x).\,\text{sen}(w.t) \tag{3}$$

$$f(x)=C_1.\text{sen}(b.x)+C_2.\cos(b.x)+C_3.\text{senh}(b.x)+C_4.\cosh(b.x) \tag{4}$$

with C_1, C_2, C_3, C_4 integration constants that depend on the boundary conditions and w the circular frequency of oscillation [rad/s], related with the parameter b by

$$w=b^2.\sqrt{EI/m} \tag{5}$$

As shown in Fig. 1, the absorber configuration here explored consists of two single span beams. Considering that in free vibration the beams oscillate with the same frequency, Eq. (5) is rewritten as

$$w=b_1^2.\sqrt{EI_1/m_1}=b_2^2.\sqrt{EI_2/m_2} \tag{6}$$

where subscript *1* is related to the main span, i.e, the principal system to be cured of harmful vibration, and subscript *2* indicates the span which is the absorber itself. These *spans* generate two sets of integration constants, so that Eq. (4) can be generalized to

$$f_j(x)=C_{1j}.\text{sen}(b_j.x)+C_{2j}.\cos(b_j.x)+C_{3j}.\text{senh}(b_j.x)+C_{4j}.\cosh(b_j.x) \tag{7}$$

Equation (7) refers to a two span beam problem with eight integration constants C_{kj} (k=1,2,3,4 and j=1,2), defined by applying eight boundary conditions.

2.1. Boundary conditions

By observing Fig.1 we can write the following boundary conditions:

1. Position A of span *1* has no translation

$$v_1(0,t)=0 \Rightarrow f_1(0)=0 \tag{8.a}$$

2. Position A of span *1* has no rotation

$$v_1^{'}(0,t)=0 \Rightarrow f_1^{'}(0)=0 \tag{8.b}$$

3. At position B, the spans have the same translation

$$v_1(L_1,t)=v_2(L_1,t) \Rightarrow f_1(L_1)-f_2(L_1)=0 \tag{8.c}$$

4. At position B, the spans have the same rotation

$$v_1^{'}(L_1,t)=v_2^{'}(L_1,t) \Rightarrow f_1^{'}(L_1)-f_2^{'}(L_1)=0 \tag{8.d}$$

5. Moment equilibrium at position B is given by

$$M_{f1}(L_1,t)-M_{f2}(L_1,t)+J_1\ddot{v}_1^{'}(L_1,t)=0 \Rightarrow EI_1f_1^{''}(L_1)-EI_2f_2^{''}(L_1)-w^2J_1f_1^{'}(L_1)=0 \tag{8.e}$$

6. Transverse force equilibrium at position B is given by

$$Q_2(L_1,t)-Q_1(L_1,t)+M_1\ddot{v}_1(L_1,t)=0 \Rightarrow m_2f_2^{'''}(L_1)-m_1f_1^{'''}(L_1)-w^2M_1f_1(L_1)=0 \tag{8.f}$$

7. Bending moment at position C, due to rotational inertia of the absorber concentrated mass, is given by

$$M_{f2}(L_1+L_2,t)=-J_2\ddot{v}_2^{'}(L_1+L_2,t) \Rightarrow EI_2f_2^{''}(L_1+L_2)-w^2J_2f_2^{'}(L_1+L_2)=0 \tag{8.g}$$

8. The transverse shear force *Q* at position C due to the translation inertia of the absorber mass

$$Q_2(L_1+L_2,t)=M_2\ddot{v}_2(L_1+L_2,t) \Rightarrow m_2f_2^{'''}(L_1+L_2)-w^2M_2f_2(L_1+L_2)=0 \tag{8.h}$$

2.2. Natural Frequencies and Mode Shapes

By introducing Eqs. (8.a)-(8.h) into the equilibrium differential equation (7), it can be recast in the following matrix form:

$$[A(l)].\{C_{11},C_{21},C_{31},C_{41},C_{12},C_{22},C_{32},C_{42}\}^T=\{0,0,0,0,0,0,0,0\}^T$$

or

$$\underset{8x8}{[A(l)]}.\underset{8x1}{\{C\}}=\underset{8x1}{\{0\}} \tag{9}$$

In Eq. (9), $l=b_1.L_1$ is the parameter that controls the existence of solutions different from the trivial null one, i.e. $(C_{kj}=0, \quad \forall k,j)$. Therefore, in order to obtain the non trivial solutions, the solutions for Eq. (9) are the eigenvalues (l_i), such that

$$\det[A(l)]=0 \Rightarrow l=l_1,l_2,...,l_i,...l_\infty \tag{10}$$

In addition, in view of Eq. (6), each root l_i furnishes a corresponding value for the natural frequencies of this double-span problem:

$$w_i[rad/s]=(l_i/L_1)^2\sqrt{EI_1/m_1} \tag{11}$$

It is interesting to point out that Eq. (11) depends only on the geometrical properties of span 1. All other properties of the problem have their influence implicitly accounted for in matrix $A(l)$.

Now, by solving Eq. (9) repeatedly for each eigenvalue l_i ($i = 1,2,3,...,n \leq \infty$), n different sets of the constants C_{kj} (k=1,2,3,4 and j=1,2) are obtained, each set corresponding to one eigenvector or mode shape. Accordingly, the combination of Eqs. (6), (7) and (11) leads to the following expression for the mode shapes

$$f_{ji}(x) = C_{1ji}.\text{sen}(b_{ji}.x) + C_{2ji}.\cos(b_{ji}.x) + C_{3ji}.\text{senh}(b_{ji}.x) + C_{4ji}.\cosh(b_{ji}.x)$$
$$\text{with} \qquad b_{ji} = (w_i^2.m_j / EI_j)^{1/4} \tag{12}$$

The explicit formulation of matrix $A(l)$ includes laborious derivatives and algebraic manipulations so that, for sake of space, it is not described here. The matrix $A(l)$ and the roots of Eq. (10) were obtained by using symbolic operations in the program MATLAB®.

2.3. Harmonic (Steady State) Response

Considering the purpose of this work, a harmonic force $p(t) = p_o.\text{sen}(\overline{w}.t)$ applied at the concentrated mass in point B, Fig. 1, excites the main system. By applying Modal Superposition (Clough, 1975) the response depends on the following generalized properties for mass, stiffness and load, respectively,

$$\tilde{m}_i = \int_0^{L_1} m_1 f_{1i}^2(x)dx + \int_{L_1}^{L_1+L_2} m_2 f_{2i}^2(x)dx + M_1 f_{1i}^2(L_1) + J_1 f_{1i}^{'2}(L_1) + ...$$
$$.......... .... + M_2 f_{2i}^2(L_1 + L_2) + J_2 f_{2i}^{'2}(L_1 + L_2) \tag{13.a}$$

$$\tilde{k}_i = \tilde{m}_i.w_i^2 \tag{13.b}$$

$$\tilde{p}_i(t) = [p_o \,\text{sen}(\overline{w}.t)].f_{1i}(L_1) = \tilde{p}_{oi}\,\text{sen}(\overline{w}.t) \quad \text{with} \quad \tilde{p}_{oi} = p_o.f_{1i}(L_1) = (m_e.e.\overline{w}^2).f_{1i}(L_1) \tag{13.c}$$

with known natural frequencies w_i and shape modes f_{ji}. The amplitude of the excitation p_o is due to the unbalanced force on the fan. It depends on the square of excitation frequency $\overline{w}$, the unbalanced mass m_e and the eccentricity e.

Now, by using the simple solution for harmonically excited single degree of freedom system (Clough 1975), the stead state response can be written as

$$v_j(x,t) = \sum_i f_{ji}(x).y_i(t) = \sum_i f_{ji}(x).r_i.\text{sen}(\overline{w}t + j_i) \tag{14.a}$$

$$r_i(\overline{w}) = \frac{\tilde{p}_{oi}}{\tilde{k}_i}.\frac{1}{\sqrt{(1-\overline{b}_i^2)^2 + (2.z_i.\overline{b}_i^2)^2}} \qquad \text{Normal response amplitude} \tag{14.b}$$

$$\overline{b}_i = \overline{w} / w_i \qquad \text{Frequency ratio} \tag{14.c}$$

$$j_i(\overline{w}) = arctg\left(\frac{2.z_i.\overline{b}_i^2}{1-\overline{b}_i^2}\right) \leq 0 \qquad \text{Normal response phase} \tag{14.d}$$

$$0 \leq z_i \leq 1 \qquad \text{Modal damping ratio} \tag{14.e}$$

Taking Eqs. (14.a)-(14.e) for $x = L_1$ and then for $x = L_1 + L_2$, after some manipulation, the response amplitudes at points B (main system) and C (absorber), respectively, are given by

$$d_{01}(\overline{w}) = \sqrt{\left(\sum_i f_{1i}(L_1).r_i(\overline{w}).\cos(j_i(\overline{w}))\right)^2 + \left(\sum_i f_{1i}(L_1).r_i(\overline{w}).\text{sen}(j_i(\overline{w}))\right)^2} \tag{15.a}$$

$$d_{02}(\overline{w}) = \sqrt{\left(\sum_i f_{2i}(L_1+L_2).r_i(\overline{w}).\cos(j_i(\overline{w}))\right)^2 + \left(\sum_i f_{2i}(L_1+L_2).r_i(\overline{w}).\mathrm{sen}(j_i(\overline{w}))\right)^2} \qquad (15.b)$$

The final expression can be obtained substituting the definitions presented in Eqs. (15), but it is a huge task and has no practical meaning. Again, by using MATLAB®, one can solve the entire problem with the equations presented in this section.

3. Results and Discussion

In order to explore the proposed model, the absorber showed in Figure 1 was conceived, according to the properties listed in Tab. 1. In the same table, the natural frequencies measured in the experimental analysis can be compared with those obtained from the proposed mathematical model and from a Finite Element simulation of the problem.

In the finite element simulation, a simple beam element was initially adopted to model the beams in order to use the same hypothesis of the analytical model. The obtained results were very similar to the analytical model, with deviations in the natural frequency not greater than 0.01 Hz, and so are not shown in Tab. 1.

The problem was also solved using solid finite elements, whose discretization can be seen in Figure 2. This element has fewer assumptions than the analytical and FEM pure beam models since distributed rotational inertia and shear deformation effects are inherent in the solid element assemblage (Bath, 1996). Accordingly, a more realistic model is reached. The results for this solid modeling are also shown in Tab. 1.

Table 1. Problem Data & Analytical (ANA), Numerical (FEM) and Experimental (EXP) Results for Natural Frequencies [Hz]

DATA (see Fig. 1)							ABSORBER MASS (M_2)(1)								
	Lj	*mj*	*Mj*	*Jj*	*Ej*	*Ij*	0.000 Kg				0.004 Kg				
j	[m]	[Kg/m]	[Kg]	[Kg.s^4]	[N/m^2]	[m^4]	MODE(2)	FEM(5)	ANA	EXP(3)	MODE(2)	FEM(5)	ANA	EXP(3)
	10^{-2}	10^{-3}	10^{-3}	10^{-5}	10^{10}	10^{-12}	1	19.71	20.70	19.4	1	18.56	20.06	18.2
1	30	322	165	26	7	424	-	-	-	-	2	24.42	30.74	26.8
2	6.1(4)	6.2	4(1)	0	21	0.05	2	142.2	166.5	-	3	150.2	166.9	-

(1): The distributed mass of the absorber beam is negligible compared with the concentrated mass, so that $M_2=0$ means that the absorber is inactive otherwise $M2\neq0$ means active

(2): With no mass M_2, the lowest resonance frequency corresponds to the system operation point. When M_2 is introduced, the first natural frequency of original system is replaced by two surrounding new natural frequencies.

(3): In the experimental analysis, the resonance frequencies are measured by using a stroboscopic light. Due to limitations of the cooler, the excitation frequency could not be larger than 40 Hz.

(4): In the experimental device, after introduction of absorber mass M_2, the length L_2 is continuously adjusted until an anti-resonance point is reached.

(5): Solid model of fig.2

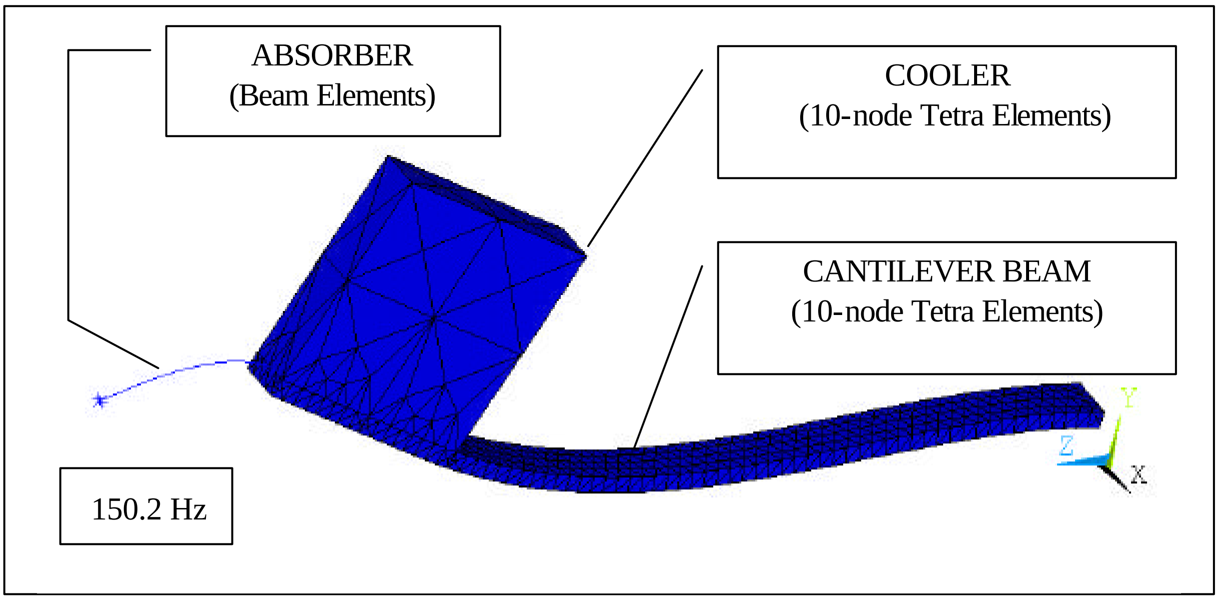

Figure 2. Finite Element Model showing Third Natural Mode Shape

The FEM harmonic solution employs the Direct Integration Method (Bath, 1996) so that, to comply with our analytical solution, some conversions of the damping properties are necessary. The coefficients of Rayleigh Damping Matrix, needed for Direct Integration, can be related to the modal damping ratios by using the orthogonal properties of mode shapes, so that

$$z_i = 0.5 * (a / w_i + b.w_i) \tag{16}$$

where a and b are the Rayleigh coefficients related to Mass and Stiffness Matrices, respectively. Now, taking the two lowest mode shapes and applying twice Eq. 6, these coefficients are given by

$$a = 2 * (\zeta_1.w_2 - \zeta_2.w_1)/(w_2 / w_1 - w_1 / w_2) \tag{17.a}$$

$$b = 2 * (\zeta_1.w_1 - \zeta_2.w_2)/(w_1^2 - w_2^2) \tag{17.b}$$

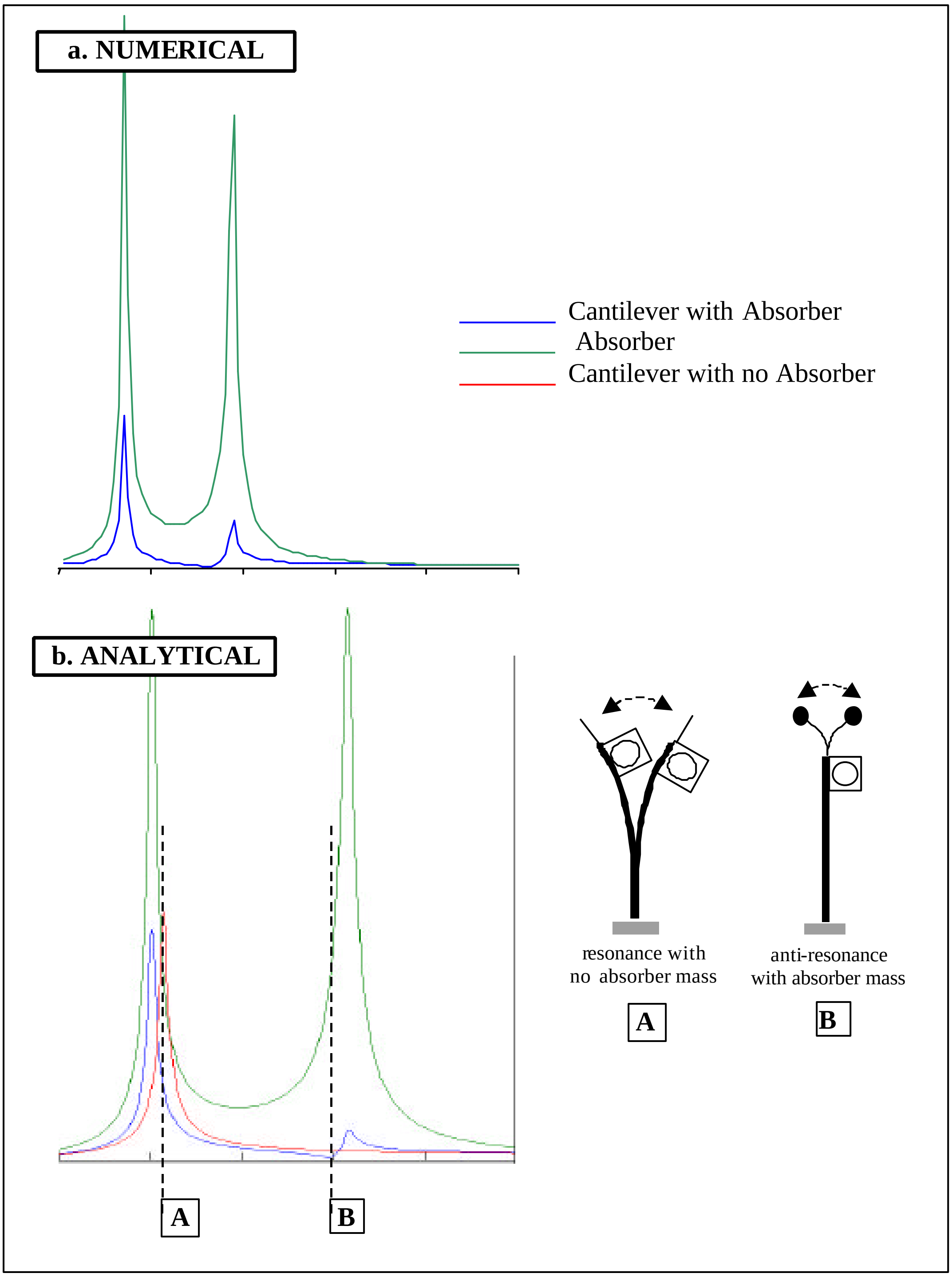

Figure 3. Solutions for Harmonic (Stead State) Response

If the same damping ratio is adopted for both mode shapes ($\zeta = \zeta_1 = \zeta_2$) and, in addition, the natural frequencies are converted to Hertz unit ($w_i\,[rad\,/\,s] = 2\pi.f_i\,[Hz]$), coefficients a and b will be defined by

$$a = 4p.z.(1/f_1 + 1/f_2) \cong 1.19166.z \qquad (18.a)$$

$$b = (z\,/\,p)\,/.(f_1 + f_2) \cong z\,/135.026 \qquad (18.b)$$

using the numerical values listed in Tab. 1.

Considering $\zeta = 1$ %, Fig. 3 shows Frequency Response Curves obtained by the FEM solid model in comparison with the analytical solution of Eq. (15) for both situations: with and without absorber. It is clear that the vibration amplitude of the main system drops drastically when absorber is activated.

The observed deviations in peaks locations are mainly due to natural frequency differences, as shown in Tab. 1. The FEM solid model fits better the experimental results due to the reasons pointed out previously. On the other hand, FEM beam and analytical models tend to give more reliable results when the length of the beams increases.

Moreover, the difference in peak magnitude can be related to damping formulations as well as to damping values. In this concern, to better approach theoretical and experimental results it should be necessary to measure vibration amplitudes and, then, to calibrate the models by varying the damping coefficients.

4. Conclusions

The analytical approach presented here for a dynamic absorber is an extension of the two-degree-of-freedom classical problem to a continuous beam system. Experimental observations and FEM results were used to validate the proposed model. The results indicate that, in case of periodic excitation when more than one harmonic component may be present in the stead state response, it is possible to use two or more absorbers, each one tuned to absorb one specific harmonic component vibration. As an extension of the problems studied here and in the work of Jalili and Olgac 1999, a future work may be focused on this direction. On the other hand, improvements of the absorber design by inserting automatic control over its length, as shown in Appendix A, makes feasible to conceive smart absorbers which would be always tuned to the vibration to be suppressed.

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6. Responsibility notice

The authors are the only responsible for the printed material included in this paper.

Appendix A. Smart Absorber

An proposed extension to the traditional absorber is here called *smart absorber*, which can be adjusted according to excitation frequency, minimizing the response amplitude of the main system for a given frequency range. In order to illustrate it, lets take a simple two-degree of freedom with no damping. Being 1 the subscript that denotes the main system and 2 the absorber, the equilibrium equation in forced harmonic vibration can be written as:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} . \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} . \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} P_o . \operatorname{sen}(\overline{w}.t) \\ 0 \end{Bmatrix} \tag{A.1}$$

Since the excitation is harmonic, the steady state response is also harmonic with frequency $\overline{\omega}$ and amplitudes X_{01} for the main system and X_{02} for the absorber. Then Eq. (A.1), solved by Direct Analytical Integration, furnishes the following result

$$\begin{Bmatrix} X_{01}(\overline{w}) \\ X_{02}(\overline{w}) \end{Bmatrix} = \begin{bmatrix} m.(\Omega_1^2 - \overline{w}^2) + \Omega_2^2 & -\Omega_2^2 \;\; ^2 \\ -\Omega_2^2 & \Omega_2^2 - \overline{w}^2 \end{bmatrix}^{-1} . \begin{Bmatrix} P_o \\ 0 \end{Bmatrix} \tag{A.2}$$

where m is the mass ratio: M_1 / M_2 and $\Omega_i = \sqrt{K_i / M_i}\,(i = 1{,}2)$ are natural frequencies of main system (i=1) and absorber (i=2). From Eq. (A.2), one can see that if the absorber tuning is adjusted so that its natural frequency coincides with the excitation frequency $(\Omega_2 = \overline{w})$, the amplitude of main system vibration is null, i.e., it remains standing for any excitation frequency and amplitude $(X_{01}(\overline{w}) = 0, \forall \overline{w}, P_o)$. On the other hand, the absorber vibrates with amplitude $X_{02}(\overline{w}) = -P_o / \overline{w}^2$, in order to apply in the main system a force constantly contrary to the excitation force.

When damping is taken is account, the simplicity in the response is unfortunately lost. However, the main conclusion remains: there is a tuning frequency for the absorber that reduces the main system amplitude to a minimum value (perhaps not a null value, but still small one).

In order to verify this conclusion, in the case of beam model of Fig. 1 excited in a given range of frequency, the equations of Section 2 can be solved for different values of absorber length L_2 and absorber mass M_2, so that we can find an optimum relation, for each frequency, that minimizes the main system vibration. As can be seen from Eq. (A.2) and Fig. 3, absorber properties are chosen in order to make the anti-resonance frequency coincides with excitation frequency.

In this context, the solution strategy can be as follow. For a given absorber mass it is necessary to look for the absorber length that minimizes the amplitude of main system, for each frequency. The curves in Fig. A.1, plotted for different masses, were obtained for the case summarized in Table 1. In this way, the principle of the smart absorber is that the length L_2 is changed with the excitation frequency according to the plotted curves, in order to mitigates the main system vibration. Of course for a practical application it is necessary to control the absorber length by an automatic mechanism.