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SENSITIVITY ANALYSIS OF VISCOELASTIC STRUCTURES

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Abstract. *In the context of sound and vibration control of mechanical systems, the use of viscoelastic materials has been regarded as a convenient strategy in many types of industrial applications. More recently, viscoelastic materiais have been combined with piezoelectric materials in a hybrid passive-active control strategy known as active constraining damping layers. In this context, numerical models based on finite element discretization have been frequently used in the analysis and design of complex structural systems incorporating viscoelastic materials. Such models must account for the typical dependency of the viscoelastic characteristics on operational and environmental parameters, such as frequency and temperature. In the analysis and optimal design of structural systems based on numerical models, sensitivity analysis is a very usefull tool. In this paper, the formulation of sensitivity analysis is developed for plates treated with constraining damping layers, considering complex frequency response functions as dynamic responses and goemetrical characteristics, such as the thicknesses of the multi-layer components, as design variables. Also, the sensitivity of the frequency responses functions with respect to temperature is introduced. As examples, response derivatives are calculated for a three-layer plate and the results obtained are compared with first-order finite-difference approximations.*

Keywords: *Viscoelasticity, Finite Elements, Plates, Sensitivity Analysis*

1. Introduction.

Passive damping approaches to the problem of vibration attenuation constitute an important subject in modern Mechanical Engineering. The use of viscoelastic materials to reduce noise and vibration levels in various types of engineering systems such as robots, automobiles, airplanes, communication satellites, buildings and space structures has been intensively investigated lately. Much of the knowledge available to date is compiled in the books by Nashif *et al.* (1985) and Mead (1998) and in some review papers such as those by Rao (2001) and Samali and Kwok (1995).

In the last two decades, a great deal of effort has been devoted to the development of mathematical models for the dynamic behavior of viscoelastic materials, accounting for its typical dependence on operational and environmental effects, such as the excitation frequency, static preloads and temperature (Christensen, 1982). Some of those models are considered to be particularly suitable to be used in combination with finite element discretization, which makes them very attractive for the modeling of complex engineering systems. Among those models, it should be mentioned the so-called Fractional Derivative Model (FDM) (Bagley and Torvik, 1979, 1983, 1985), the Golla-Hughes-McTavish Model (GHM) (Golla and Hughes, 1985; McTavish and Hughes, 1993) and the Anelastic Displacement Fields Model (ADF) suggested by Lesieutre and co-workers (Lesieutre, 1992; Lesieutre and Bianchini, 1995; Lesieutre and Lee, 1996). These models can represent the viscoelastic behavior in the frequency domain. Moreover, the GHM and ADF models can also provide convenient time domain representations. For themorheologically-simple viscoelastic materials, the influence of temperature can be accounted for based on the concept of reduced frequency, according to the relation proposed by Drake and Soovere (1984).

In the context of analysis and design of structural systems, an important topic to be addressed is the so-called sensitivity analysis, which enables to evaluate the effect of variations of physical and/or geometrical parameters on the mechanical behavior. Moreover, sensitivity analysis constitutes an important step in various types of problems such as model updating, optimal design, system identification and control.

The sensitivity analysis is based on the evaluation of the derivatives (most frequently limited to the first order) of the system response with respect to a set of parameters of interest. It can be associated to different kinds of mechanical responses: static displacements, eigenvalues and eigenvectors, frequency response functions and time responses (Haug *et al.*, 1986). According to Murthy and Haftka (1988), the optimal design structural systems has a narrow connection with sensitivity analysis, since a significant part of typical optimization algorithms generally perform a large number of evaluations of the system response for different values of the design variables. Derivatives can be used to approximate the response of modified systems, thus reducing the cost of re-analysis, specially for high-order systems.

Several approaches are available for performing sensitivity analysis of dynamic responses, such as the Modal Method (Fox and Kapoor, 1968), Nelson's Method (Nelson, 1976) and the Improved Modal Method (Lim *et al.*, 1987). However, applications to the case of structural systems containing viscoelastic components are not numerous.

In this remainder, sensitivity analysis is applied to plates treated with surface constraining damping layer treatments, considering, as dynamic responses, complex frequency response functions. As design parameters, goemetrical characteristics, such as the thicknesses of the multi-layer components are considered. Also, an approach for evaluating the sensitivity of the frequency responses functions with respect to temperature is introduced. To illustrate the use of the formulation developed, response derivatives are calculated for a three-layer sandwich plate and the results obtained are compared with first-order finite-difference approximations.

2. The complex modulus approach.

According to the linear theory of viscoelasticity (Christensen, 1982), the one-dimensional stress-strain relation can be expressed, in Laplace domain, as follows:

$$\sigma(s) = G(s)\varepsilon(s) \tag{1}$$

where:

$$G(s) = G_r + H(s) \tag{2}$$

In the equation above, G_r is the *static modulus*, representing the elastic behavior and $H(s)$ is the *relaxation function*, associated to the dissipation effects. When evaluated along the imaginary axes of the s-plane $(s = i\omega)$, the complex modulus and the capacity of dissipation of the viscoelastic materials, are expressed in the following forms:

$$G(\omega) = G'(\omega) + iG''(\omega) \tag{3}$$

$$\eta(\omega) = G''(\omega)/G'(\omega) \tag{4}$$

where $G'(\omega)$, $G''(\omega)$ and $\eta(\omega)$ are defined, respectively, as *storage modulus, loss modulus* and *loss factor* of the viscoelastic materials.

3. Influence of temperature on the viscoelastic behavior.

According to Nashif *et al.* (1985), the temperature is usually considered to be the single most important environmental factor affecting the properties of viscoelastic materials. This makes it important to account for this factor in the modeling of structural systems containing viscoelastic elements. This is possible by making use of the so-called *Frequency-Temperature Superposition Principle - FTSP*, which establishes an relation between the effects of the excitation frequency and temperature on the proprieties of the thermorheologically simple viscoelastic materials. This relation leads to a function of the temperature known as the *shift factor*, which implies that the viscoelastic behavior at different temperatures can be related to each other by changes (or shifts) in the actual values of the excitation frequency. This leads to the concept of *reduced frequency* (Christensen, 1982). Symbolically, the FTSP can be expressed under the following forms:

$$G(\omega,T) = G(\omega_r, T_z) = G(\alpha_T\omega, T_z) \tag{5}$$

$$\eta(\omega_r, T) = \eta(\alpha_T\omega, T_z) \tag{6}$$

where $\omega_r = \alpha_T(T)\omega$ is the reduced frequency, ω is the actual frequency, $\alpha_T(T)$ is the shift function that must be identified from experimental tests for a specific type of viscoelastic material, and T_z is a reference value of temperature.

Figure 1 illustrates the FTSP, showing that having the modulus and loss factor of an arbitrary viscoelastic material for different temperatures T_{-2}, T_{-1}, T_0, T_1 and T_2 (where the curve represented by T_0 is taken as a reference), if convenient horizontal shifts along the frequency axis is applied to each of these curves, they can be combined into a single one. The horizontal shift α_T depends on the temperature.

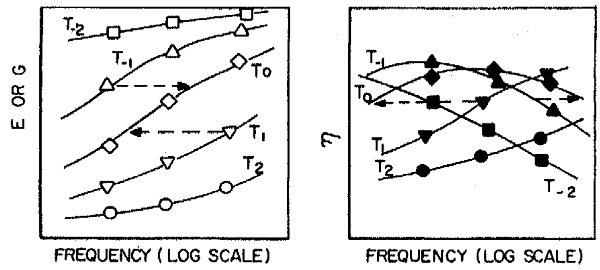

Figure 1. Illustration of the Frequency-Temperature Superposition Principle (adapted from Nashif *et al.* (1985)).

4. Viscoelastic behavior incorporated into finite element models.

The finite element equations of motion of viscoelastic structure containing N degrees of freedom can be expressed as follows:

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\boldsymbol{K}(\omega,T)\boldsymbol{q}(t)=\boldsymbol{f}(t) \tag{7}$$

where $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{K}(\omega,T) \in R^{NxN}$ are the mass (symmetric, positive-definite), damping (symmetric, nonnegative-definite), and stiffness (symmetric, nonnegative-definite) matrices. $\boldsymbol{q}(t), \boldsymbol{F}(t) \in R^N$ are the vectors of displacement and external loads, respectively.

It is assumed that the structure contains both elastic and viscoelastic elements, so that the stiffness matrix can be decomposed as follows:

$$\boldsymbol{K}(\omega,T)=\boldsymbol{K}_e+\boldsymbol{K}_v(\omega,T) \tag{8}$$

where $\boldsymbol{K}_e$ is the stiffness matrix corresponding to the purely elastic substructure and $\boldsymbol{K}_v(\omega,T)$ is the stiffness matrix associated with the viscoelastic substructure. The inclusion of the frequency-dependent behavior of the viscoelastic material can be made by using the so-called *elastic-viscoelastic correspondence principle* (Christensen, 1982), according to which, for a given temperature, $\boldsymbol{K}_v(\omega,T)$ is first generated for specific types of elements (rods, beam, plates, etc.) assuming that the longitudinal modulus $E(\omega,T)$ and/or the shear modulus $G(\omega,T)$ (according to the stress-state) are constant, independent on frequency. After the finite element matrix is constructed, such moduli are made frequency-dependent according to a particular viscoelastic constitutive model adopted. Moreover, by assuming a constant Poisson ratio for the viscoelastic material, $E(\omega,T)$ becomes proportional to $G(\omega,T)$ through the relation $G(\omega,T)=E(\omega,T)/2(1+\nu)$. Then, of the two moduli can be factored-out of the stiffness matrix of the viscoelastic sub-structures. By writing:

$$\boldsymbol{K}_v(\omega,T)=G(\omega,T)\overline{\boldsymbol{K}}_v \tag{9}$$

equations (7), (8) and (9) are combined to give:

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\left(\boldsymbol{K}_e+G(\omega,T)\overline{\boldsymbol{K}}_v\right)\boldsymbol{q}(t)=\boldsymbol{f}(t) \tag{10}$$

The difficulty in dealing with Eq. (10) for predicting the vibration response in the time domain and performing eigenvalue analysis commes from the fact that the stiffness matrix is not constant and depends on frequency. Some procedures for overcoming such difficulty have been suggested, based on the adoption of particular representations for the frequency-dependent behavior of the viscoelastic materials. Such an approach is used in the Fractional Derivative, Golla-Hughes-McTavish and Anelastic Displacement Field models (see references in Section 1), which enable to transform Eq. (10) into state-space equations of motion with constant (frequency-independent) state matrices. Equation (10) can be directly used for calculating the steady-state harmonic responses in the frequency domain, by assuming:

$$\boldsymbol{f}(t)=\boldsymbol{F}\,e^{i\omega t} \tag{11.a}$$

$$\boldsymbol{q}(t)=\boldsymbol{Q}(\omega)e^{i\omega t} \tag{11.b}$$

Upon introduction of Eqs. (11) into Eq. (10), the following relation is obtained between the amplitudes of the excitation forces and the amplitudes of the harmonic responses:

$$\boldsymbol{Q}(\omega) = \boldsymbol{H}(\omega,T)\boldsymbol{F} \tag{12}$$

where the so-named frequency response function (FRF) matrix is expressed as:

$$\boldsymbol{H}(\omega,T) = \left[-\omega^2\boldsymbol{M} + i\omega\boldsymbol{C} + \boldsymbol{K_e} + G(\omega,T)\overline{\boldsymbol{K}}_{\boldsymbol{v}}\right]^{-1} \tag{13}$$

In this paper, based on Eq. (13), we consider the sensitivity of FRFs with respect to physical and/or geometrical parameters featuring in the finite element matrices and also with respect to the temperature.

5. A three-layer sandwich plate finite element.

In this section the formulation of a three-layer sandwich plate finite element is summarized based on the original development made by Kathua and Cheung (1973). Figure 2 depicts a rectangular element formed by an elastic base-plate, a viscoelastic core and an elastic constraining layer, whose dimensions in directions x and y are denoted by a and b, respectively. In the remainder, superscripts (1), (2) and (3) are used to identify the quantities pertaining the base-plate, the viscoelastic core and the constraining layer, respectively.

Space discretization is made by considering 4 nodes and 7 degrees-of-freedom per node, representing the nodal longitudinal displacements of the base-plate middle plane in directions x and y (denoted by u_1 e v_1), the nodal longitudinal displacements of the constraining layer middle plane in directions x and y (denoted by u_3 e v_3), the nodal transverse displacement, w, and the nodal cross-section rotations of the layers about axes x and y, denoted by θ_x and θ_y, respectively. The vectors of nodal and element degrees-of-freedom are then expressed as follows:

$$\boldsymbol{\delta^i} = \left[u_1^i \; v_1^i \; u_3^i \; v_3^i \; w^i \; \theta_x^i \; \theta_y^i\right]^T \; (i = 1,2,3,4), \quad \boldsymbol{\delta} = \left[\boldsymbol{\delta}^{1^T} \; \boldsymbol{\delta}^{2^T} \; \boldsymbol{\delta}^{3^T} \; \boldsymbol{\delta}^{4^T}\right]^T \tag{14}$$

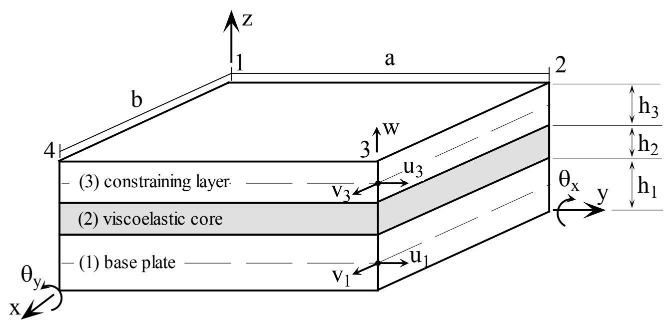

Figure 2. Three-layer sandwich plate finite element.

The following assumptions are adopted:

- normal stresses and strains in direction z are neglected for all the three layers.
- the elastic layers (base-plate and constraining layer) are modeled according to Kirchhoff's theory which neglects the effects associated to transverse (thickness-wise) shear;
- for the the viscoelastic core, Mindlin's theory is adopted, which includes transverse shear;
- the transverse displacement w is the same for all the three layers and cross-section rotations θ_x and θ_y are assumed to be the same for both the elastic layers.

By imposing kinematic relations that enforce the continuity of displacements along the interfaces between the layers, the following expressions are found for the longitudinal displacements of the viscoelastic core middle plane in directions x and y in terms of previously defined nodal coordinates:

$$u_2 = \frac{1}{2}\left[u_1 + u_3 + \frac{h_1 - h_3}{2}\frac{\partial w}{\partial x}\right], \quad v_2 = \frac{1}{2}\left[v_1 + v_3 + \frac{h_1 - h_3}{2}\frac{\partial w}{\partial y}\right] \tag{15}$$

The longitudinal and transverse displacements are interpolated within the element as follows:

$$
\begin{aligned}
&u_1 = u_1(x,y) = a_1 + a_2 x + a_3 y + a_4 xy \qquad u_3 = u_3(x,y) = a_9 + a_{10} x + a_{11} y + a_{12} xy \\
&v_1 = v_1(x,y) = a_5 + a_6 x + a_7 y + a_8 xy \qquad v_3 = v_3(x,y) = a_{13} + a_{14} x + a_{15} y + a_{16} xy \\
&w = w(x,y) = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2 + b_7 x^3 + b_8 x^2 y + b_9 xy^2 + b_{10} y^3 + b_{11} x^3 y + b_{12} xy^3
\end{aligned}
\tag{16}
$$

Based on the hypotheses of the stress-states assumed for each layer, the following stress-strain relations apply:

- For the base-plate:

$$
\begin{Bmatrix} \sigma_x^{(1)} \\ \sigma_y^{(1)} \\ \tau_{xy}^{(1)} \end{Bmatrix} = \begin{bmatrix} \dfrac{E^{(1)}}{1-\nu^{(1)2}} & \dfrac{E^{(1)}\nu^{(1)}}{1-\nu^{(1)2}} & 0 \\ \dfrac{E^{(1)}\nu^{(1)}}{1-\nu^{(1)2}} & \dfrac{E^{(1)}}{1-\nu^{(1)2}} & 0 \\ 0 & 0 & G^{(1)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{(1)} \\ \varepsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \text{ or } \boldsymbol{\sigma}^{(1)} = \boldsymbol{E}^{(1)}\boldsymbol{\varepsilon}^{(1)} = \boldsymbol{E}^{(1)}\boldsymbol{D}^{(1)}\boldsymbol{\delta} \tag{17}
$$

- For the viscoelastic core:

$$
\begin{Bmatrix} \sigma_x^{(2)} \\ \sigma_y^{(2)} \\ \tau_{xy}^{(2)} \\ \tau_{xz}^{(2)} \\ \tau_{yz}^{(2)} \end{Bmatrix} = \begin{bmatrix} \dfrac{E^{(2)}(\omega,T)}{1-\nu^{(2)2}} & \dfrac{E^{(2)}(\omega,T)\nu^{(2)}}{1-\nu^{(2)2}} & 0 & 0 & 0 \\ \dfrac{E^{(2)}(\omega,T)\nu^{(2)}}{1-\nu^{(2)2}} & \dfrac{E^{(2)}(\omega,T)}{1-\nu^{(2)2}} & 0 & 0 & 0 \\ 0 & 0 & G^{(2)}(\omega,T) & 0 & 0 \\ 0 & 0 & 0 & G^{(2)}(\omega,T) & 0 \\ 0 & 0 & 0 & 0 & G^{(2)}(\omega,T) \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{(2)} \\ \varepsilon_y^{(2)} \\ \gamma_{xy}^{(2)2} \\ \gamma_{xx}^{(2)2} \\ \gamma_{yz}^{(2)2} \end{Bmatrix}, \text{ or}
$$

$$
\boldsymbol{\sigma}^{(2)} = \boldsymbol{E}^{(2)}(\omega,T)\boldsymbol{\varepsilon}^{(2)} = \boldsymbol{E}^{(2)}(\omega,T)\boldsymbol{D}^{(2)}\boldsymbol{\delta} \tag{18}
$$

- For the constraining layer:

$$
\begin{Bmatrix} \sigma_x^{(3)} \\ \sigma_y^{(3)} \\ \tau_{xy}^{(3)} \end{Bmatrix} = \begin{bmatrix} \dfrac{E^{(3)}}{1-\nu^{(3)2}} & \dfrac{E^{(3)}\nu^{(3)}}{1-\nu^{(3)2}} & 0 \\ \dfrac{E^{(3)}\nu^{(3)}}{1-\nu^{(3)2}} & \dfrac{E^{(3)}}{1-\nu^{(3)2}} & 0 \\ 0 & 0 & G^{(3)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^{(3)} \\ \varepsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \text{ or } \boldsymbol{\sigma}^{(3)} = \boldsymbol{E}^{(3)}\boldsymbol{\varepsilon}^{(3)} = \boldsymbol{E}^{(3)}\boldsymbol{D}^{(3)}\boldsymbol{\delta} \tag{19}
$$

where matrices $\boldsymbol{D}^{(k)}$ $(k = 1,2,3)$ are formed by derivatives of the shape functions according to the differential operators appearing in the strain-displacement relations for each layer.

Based on the stress-strain relations above, neglecting viscous damping, the strain and kinetic energies of the composite plate element are formulated. Lagrange equations are then used, considering the nodal displacements and rotations as generalized coordinates, to obtain the element stiffness and mass matrices. Such procedure has been fully developed by Stoppa (2003) using symbolic computation.

According to Eqs. (8) and (9), the element stiffness matrix can be expressed as follows:

$$
\boldsymbol{K}_e = \boldsymbol{K}^{(1)} + \boldsymbol{K}^{(3)}, \ \boldsymbol{K}_v(\omega,T) = \boldsymbol{K}^{(2)}(\omega,T) = G(\omega,T)\overline{\boldsymbol{K}}^{(2)} \tag{20}
$$

where $\boldsymbol{K}^{(1)}$, $\boldsymbol{K}^{(3)}$ and $\boldsymbol{K}^{(2)}(\omega,T)$ are, respectively, the element stiffness matrices of the base-plate, constraining layer and viscoelastic core.

The assembling of element matrices into global matrices follows the standard procedure based on the enforcement of connectivity expressing rotation and displacement continuity between neighborg elements through common nodes (Maia *et al.*, 1997).

6. Sensitivity analysis of structural responses.

The global finite element matrices, $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}(\omega,T)$ appearing in Eq. (7), establish the dependency of the response of the system with respect to a set of design parameters, which include physical and geometrical characteristics and also the temperature. Such functional dependency can be expressed in a general manner as follows:

$$\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{M}(\boldsymbol{p}), \boldsymbol{C}(\boldsymbol{p}), \boldsymbol{K}(\boldsymbol{p})) \tag{21}$$

where $\boldsymbol{r}$ and $\boldsymbol{p}$ designate vectors of structural responses and design parameters, respectively.

The sensitivity of the structural responses with respect to a given parameter p_i, evaluated for a prescribed set of design parameter $\boldsymbol{p}^0$ is defined as the following partial derivative:

$$\left.\frac{\partial \boldsymbol{r}}{\partial p_i}\right|_{p^0} = \lim_{\Delta p_i \to 0}\left[\frac{\boldsymbol{r}\left(\boldsymbol{M}\left(p_i^0+\Delta p_i\right), \boldsymbol{C}\left(p_i^0+\Delta p_i\right), \boldsymbol{K}\left(p_i^0+\Delta p_i\right)\right)}{\Delta p_i} - \frac{\boldsymbol{r}\left(\boldsymbol{M}\left(p_i^0\right), \boldsymbol{C}\left(p_i^0\right), \boldsymbol{K}\left(p_i^0\right)\right)}{\Delta p_i}\right] \tag{22}$$

where Δp_i is a variation applied to the initial value parameter p_i^0, while all other parameters remain unchanged.

The sensitivity with respect to a given parameter p_i can be estimated by finite differences by computing successively the responses corresponding to $p_i = p_i^0$ and $p_i = p_i^0 + \Delta p_i$ and then computing:

$$\left.\frac{\partial \boldsymbol{r}}{\partial p_i}\right|_{p_i^0} \approx \left[\frac{\boldsymbol{r}\left(\boldsymbol{M}\left(p_i^0+\Delta p_i\right), \boldsymbol{C}\left(p_i^0+\Delta p_i\right), \boldsymbol{K}\left(p_i^0+\Delta p_i\right)\right)}{\Delta p_i} - \frac{\boldsymbol{r}\left(\boldsymbol{M}\left(p_i^0\right), \boldsymbol{C}\left(p_i^0\right), \boldsymbol{K}\left(p_i^0\right)\right)}{\Delta p_i}\right] \tag{23}$$

Such approach is in general not efficient from the computation point of view. Moreover, the results depend upon the choice of the value of the parameter increment Δp_i.

Another strategy consists in computing the analytical derivatives of the structural responses with respect to the parameters of interest. This approach is considered in the following sections.

6.1. Sensitivity of frequency response functions with respect to structural parameters.

Consider the FRF matrix of a viscoelastically damped system as given by Eq. (13). Sensitivity with respect to a given structural parameter can be computed by deriving the relation $\boldsymbol{H}(\omega, p)\boldsymbol{H}^{-1}(\omega, p) = \boldsymbol{I}$, which leads to the following expression:

$$\left.\frac{\partial \boldsymbol{H}(\omega,T,p)}{\partial p_i}\right|_{(\omega,T^0,p^0)} = -\boldsymbol{H}\left(\omega,T^0,p^0\right)\left(-\omega^2\frac{\partial \boldsymbol{M}\left(p^0\right)}{\partial p_i} + i\,\omega\frac{\partial \boldsymbol{C}\left(p^0\right)}{\partial p_i} + \frac{\partial \boldsymbol{K}\left(\omega,T^0,p^0\right)}{\partial p_i}\right)\boldsymbol{H}\left(\omega,T^0,p^0\right) \tag{24}$$

Regarding the equation above, it should be noted that when parameter p_i appears explicitly in matrices $\boldsymbol{M}$ and/or $\boldsymbol{C}$ and/or $\boldsymbol{K}$ (and this is frequently the case), the derivatives of these matrices with respect to such parameter is straightforward.

6.2. Sensitivity of the frequency response functions with respect of temperature.

The computation of the derivatives of FRFs with respect to temperature requires that such paramter appear explicity in the stiffness matrix of the viscoelastic substructure. With this aim, a procedure suggested in Section 3, based on the use of the *frequency-temperature equivalence principle* and the concept of reduced frequency can be used for the thermorheologically simple viscoelastic materials.

By combining Eqs. (5) and (13) with (24), one writes:

$$\boldsymbol{H}(\omega,T) = \left[-\omega^2\boldsymbol{M} + i\omega\boldsymbol{C} + \boldsymbol{K}_e + \boldsymbol{G}(\omega_r,T_z)\overline{\boldsymbol{K}}_v\right]^{-1} \tag{25}$$

$$\left.\frac{\partial \boldsymbol{H}(\omega,T,p)}{\partial T}\right|_{(\omega,T^0,p^0)} = -\boldsymbol{H}\left(\omega_r,T_z,p^0\right)\left[\frac{\partial G(\omega_r,T_z)}{\partial T}\overline{\boldsymbol{K}}_v\left(p^0\right)\right]\boldsymbol{H}\left(\omega_r,T_z,p^0\right) \tag{26}$$

Based on relations (5) and (6), the derivative of the complex modulus with respect to the temperature can be computed as follows:

$$\frac{\partial G(\omega_r, T_z)}{\partial T} = \frac{\partial G}{\partial \omega_r}\frac{\partial \omega_r}{\partial T} = \frac{\partial G}{\partial \omega_r}\frac{\partial \alpha_T}{\partial T}\omega \tag{27}$$

where the functions $G(\omega_r)$ and $\alpha_T(T)$ are given for a specific viscoelastic material.

7. Numerical applications to plates treated with passive constraining damping layer.

To illustrate the procedure for computation of the sensitivity of FRFs with respect to thickness and temperature, numerical tests were performed on the FE model of a freely suspended plate made of aluminum, fully treated with a constraining damping layer made of a layer of 3M™ ISD112 viscoelastic material and an outer aluminum sheet as shown in Fig. 3. The FE model, whose physical and geometrical characteristics are given in Tab. 1, has a total number of 80 finite elements, 99 nodes and 693 degrees-of-freedom. The computations consisted in obtaining the sensitivities of the driving point FRFs corresponding to the point I indicated on Fig. 3, denoted by $\boldsymbol{H}_{II}(\omega, T, p)$.

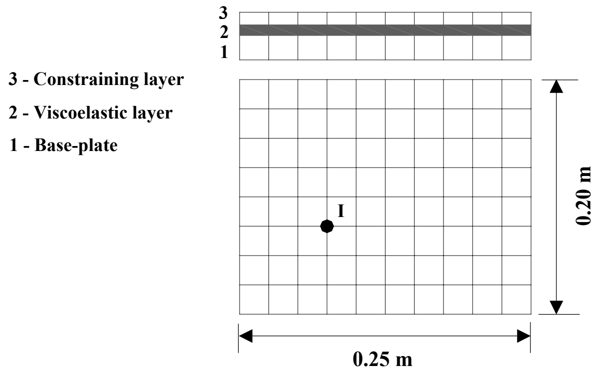

Figure 3. Illustration of the FE model of a free plate with full surface treatment.

Table 1. Physical and geometrical properties of FE model of the three-layer plate.

Layer	Thickness [m]	Modulus [N/m^2]	Poisson ratio	Density [kg/m^3]
Constraining	$0.5\text{x}10^{-3}$	$70.3\text{x}10^{9}$	0.345	2750
Viscoelastic	$20.0\text{x}10^{-5}$	---	0.500	1099.5
Base-plate	$3.0\text{x}10^{-3}$	$70.3\text{x}10^{9}$	0.345	2750

As the result of a comprehensive experimental work, Drake and Soovere (1984) proposed the following analytical expressions to represent the complex modulus and the shift factor as functions of the reduced frequency for temperatures in the interval [210K – 360K] and frequencies in the interval [1Hz – 1x10^{6}Hz], for the 3M™ISD112 viscoelastic material:

$$G(\omega_r) = B_1 + B_2 \Big/ \left(1 + B_5 (i\omega_r / B_3)^{-B_6} + (i\omega_r / B_3)^{-B_4}\right) \tag{28}$$

$$log(\alpha_T) = a\left(\frac{1}{T} - \frac{1}{T_z}\right) + 2.303\left(\frac{2a}{T_z} - b\right)log\left(\frac{T}{T_z}\right) + \left(\frac{b}{T_z} - \frac{a}{T_z^2} - S_{AZ}\right)(T - T_z) \tag{29}$$

where $B_1 = 0.4307\,MPa$, $B_2 = 1200\,MPa$, $B_3 = 0.1543\,MPa$, $B_4 = 0.6847$, $B_5 = 3.2410$ $B_6 = 0.1800$.

$$T_z = 290\,K,\; T_L = 210\,K,\; T_H = 360\,K,\; S_{AZ} = 0.05956\,K^{-1},\; S_{AL} = 0.1474\,K^{-1},\; S_{AH} = 0.009725\,K^{-1},$$

$$a = \left(\frac{D_B C_C - C_B D_C}{D_E}\right),\; b = \left(\frac{D_C C_A - C_C D_A}{D_E}\right),\; C_A = \left(\frac{1}{T_L} - \frac{1}{T_z}\right)^2,\; C_B = \left(\frac{1}{T_L} - \frac{1}{T_z}\right),\; C_C = S_{AL} - S_{AZ},$$

$$D_A = \left(\frac{1}{T_H} - \frac{1}{T_z}\right)^2,\; D_B = \left(\frac{1}{T_H} - \frac{1}{T_z}\right),\; D_C = S_{AH} - S_{AZ},\; D_E = D_E C_A - D_A C_B.$$

The derivatives of Eqs. (28) and (29) with respect to ω_r and T , are given by the following expressions:

$$\frac{\partial G}{\partial \omega_r} = B_2 B_3^{B_6}\left(B_5 B_6 e^{-\frac{iB_6\pi}{2}} (1/\omega_r)^{B_6+1} + B_4 e^{-\frac{iB_4\pi}{2}} (1/\omega_r)^{B_4+1} \right) \Bigg/ \left(1 + B_5 e^{-\frac{iB_6\pi}{2}} (B_3/\omega_r)^{B_6} + e^{-\frac{iB_4\pi}{2}} (B_3/\omega_r)^{B_4} \right)^2 \quad (30)$$

$$\frac{\partial \alpha_T}{\partial T} = \alpha_T \left(-\frac{a}{T^2} + 2.303\left(\frac{2a}{T_z} - b\right)\frac{loge}{T} + \frac{b}{T_z} - \frac{a}{T_z^2} - S_{AZ} \right) ln\,10 \quad (31)$$

For the purpose of illustration, Fig. 4 depicts the curves representing the variations of the storage modulus, loss modulus and loss factor as functions of the reduced frequency, as obtained from Eq. (28), and the plot of the shift factor as a function of the temperature, as given by Eq. (29).

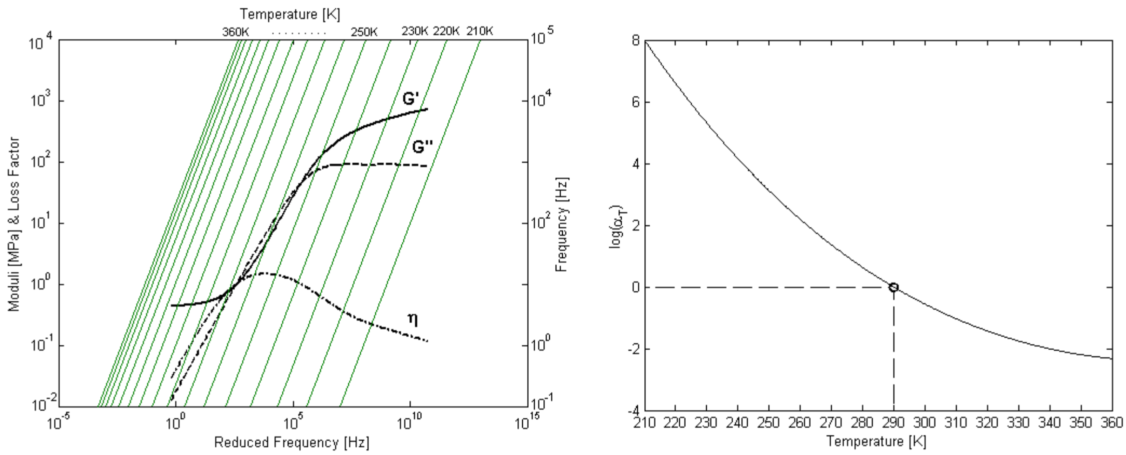

Figure 4. Master and shift factor curves for the 3M™ISD112 viscoelastic material.

7.1. Sensitivity of FRF with respect to structural parameters.

In this example, the thicknesses of the constraining and viscoelastic layers were considered as design variables in the computation of the sensitivities of the FRF $\boldsymbol{H}_{II}(\omega,T,p)$. The results obtained by using the first-order derivatives according to Eq. (24), as compared to the corresponding values calculated by finite differences (having been adopted a variation from 2.5% of the nominal values of the parameters given in Table 1), are shown in Fig. 5. In the same figure, the real and imaginary parts of the FRF $H_{II}(\omega,T,p)$, multiplied by convenient scale factors, are also shown. This figure enables to evaluate the accuracy of the computed derivatives, as demonstrated by their agreement with the results obtained by finite differences. In addition, based on the amplitudes of the sensitivity functions one can draw important information about the degree of influence of the design variables upon the amplitudes of the FRF and, in particular, on the resonance peaks.

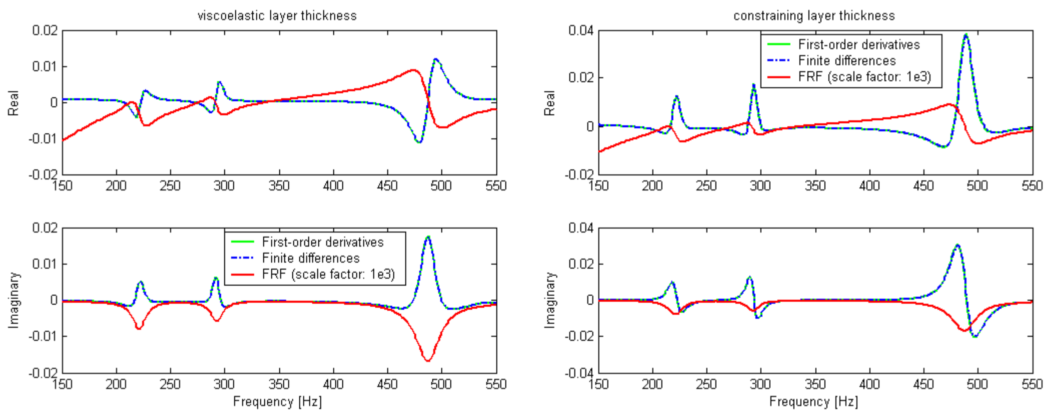

Figure 5. FRF sensitivities with respect to the thicknesses of the viscoelastic and constraining layers.

Figure 6 shows the real and imaginary parts of the sensitivities of the FRF with respect to temperature, for two different values of the nominal temperature, as compared to their counterparts calculated by finite differences, using variations of 2.0% of the nominal temperature values. From these figures, it can be concluded that the first-order derivatives compare fairly well with finite differences and enable to evaluate the degree of influence of temperature changes within the frequency band of interest.

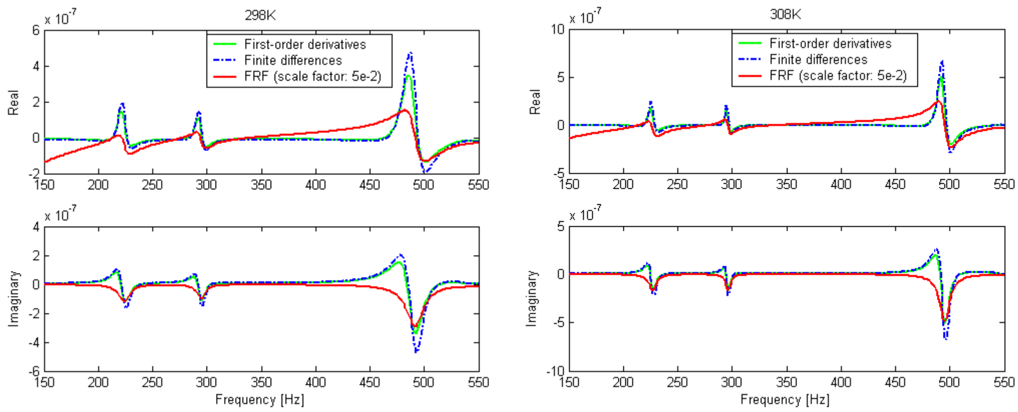

Figure 6. Sensitivities of the FRF with respect to temperatures for nominal temperature values 298K and 308K.

As a complementary demonstration of the utility of first-order derivatives, Figure 7 enables to compare the FRF amplitudes computed by two different forms: in the first, from a nominal value of the design variable (viscoelastic layer thickness, constraining layer thickness and temperature), variations are voluntarily given to such values (1.5% for both layer thicknesses and for temperature). Then, the exact FRFs of the "perturbed" system are computed. In the second form, the FRFs of the "perturbed" system are estimated from the FRFs of the nominal system by using the first order derivatives through first-order Taylor series expansions. As can be seen, the exact FRFs, represented by red lines, are fairly accurately approximated by the series expansions, represented by blue lines. The agreement is less satisfactory for the temperature variations, leading to conclude that first-order derivatives might not be accurate enough to predict variations of the dynamic behavior associated to large variations of the temperature.

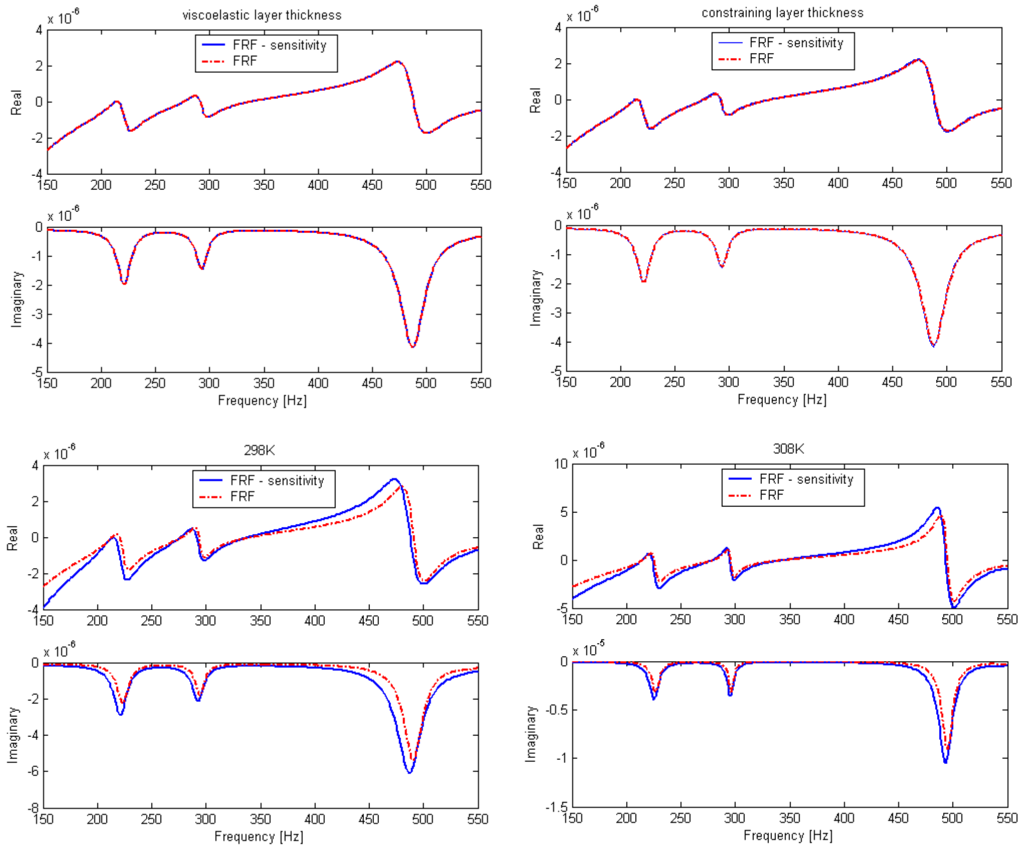

Figure 7 – Exact and first-order approximations of the FRFs of the system perturbed by variations of the design variables

8. Concluding remarks.

In this paper, sensitivity analysis based on finite element models of structural systems containing viscoelastic materials has been addressed. A formulation has been developed for the computation of first-order derivatives of frequency response functions with respect to two different kinds of parameters, namely: a) physical and/or geometrical structural parameters which appear explicitly in the finite element matrices; b) the environmental parameter temperature. Applications have been done to rectangular plates fully treated with constraining layers.

It can be concluded that the use of the complex modulus approach, combined with the concepts of shift factor and reduced frequency - justified by the principle of superposition frequency-temperature - has shown to be an adequate strategy to account for the typical dependency of the viscoelastic characteristics with respect to frequency and temperature in the finite element models of complex structural systems.

As illustrated in the numerical applications presented, the sensitivities of complex frequency response functions convey valuable information about the influence of the design parameters on the dynamic behavior of the system, being also a very useful tool for the design, performance analysis and optimization of viscoelastically damped systems.

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10. References.

Bagley, R. L. and Torvik, P. J., 1979, "A Generalized Derivative Model for an Elastomer Damper", The Shock and Vibration Bulletin, No. 49, Pt. 2, pp 135-143.

Bagley, R. L. and Torvik, P. J., 1983, "Fractional Calculus - A Different Approach to the Analysis of Viscoelastically Damped Structures", AIAA Journal, Vol. 21, No. 5, pp 741-748.

Bagley, R. L. and Torvik, P. J., 1985, "Fractional Calculus in the Transient Analysis of Viscoelastically Damped Structures", AIAA Journal, Vol. 23, No. 6, pp 918-925.

Collins, J. D., Hart, G. C., Hasselman, T. K. and Kennedy, B., 1974, "Statistical Identification of Structures", AIAA J., Vol. 12.

Christensen, R. M., 1982, "Theory of Viscoelasticity: An Introduction", Academic Press, Inc., New York, 2nd edition.

Drake, M. L. and Soovere J., 1984, "A design guide for damping of aerospace structures", AFWAL Vibration Damping Workshop Proceedings 3.

Fox, R. L. and Kapoor, M. P.; 1968, "Rates of Change of Eigenvalues and Eigenvectors", AIAA J., Vol. 6.

Golla, D. F. and Hughes, P. C., 1985, "Dynamics of Viscoelastic Structures - A Time Domain, Finite Element Formulation", Journal of Applied Mechanic, Vol 52, pp 897-906.

Khatua, T. P. and Cheung, Y. K., 1973, "Bending and Vibration of Multilayer Sandwich Beams and Plates", International Journal for Numerical Methods in Engineering, Vol. 6, pp 11-24.

Lesieutre, G. A., 1992, "Finite Element for Dynamic Modeling of Uniaxial Rods with Frequency dependent Material Properties", International Journal of Solids and Structures., Vol. 29, No. 12, pp. 1567-1579.

Lesieutre, G. A. and Bianchini, E., 1995, "Time Domain Modeling of Linear Viscoelasticity Using Anelastic Displacement Fields", Journal of Vibration and Acoustics, Transations of the ASME, Vol. 117, pp 424-430.

Lesieutre, G. A. and Lee, U., 1996, "A Finite Element for Beams Having Segmented Active Constrained Layers With Frequency-dependent Viscoelastics", Smart Materials & Structures, Vol 5, Iss 5, pp 615-627.

Lim, K. B., Junkins, J. L. and Wang, P. B., 1987, "Re-examination of Eigenvector Derivatives", AIAA J., Vol. 10.

MacTavish, D. and Hughes, P. C., 1993, "Modeling of Linear Viscoelastic Space Structures", Journal of Vibration and Acoustics, Vol 115, pp 103-115.

Maia, N. M. M. and Montalvão e Silva, J. M., 1997, "Theoretical and Experimental Modal Analysis", Research Studies Press LTD., England, pp. 468.

Mead, D. J., 1998, "Passive Vibration Control", Wiley, Canada, pp. 554.

Murthy, D. V. and Haftka, R. T., 1988, "Derivatives of Eigenvalues and Eigenvectors of a General Complex Matrix", International Journal for Numerical Methods in Engineering, Vol. 26, pp 293- 311.

Nashif, A. D., Jones, D. I. G. and Henderson, J. P., 1985, "Vibration Damping", John Wiley & Sons, New York.

Nelson, R. B., 1976, "Simplified Calculation of Eigenvector Derivatives", AIAA J., Vol. 14.

Prasad, B. and Emerson, J. F., 1982, "A General Capability of Design Sensitivity for Finite Element Systems", AIAA/ASME/ASCE/AHA 23rd Structures, Structural Dynamics and Material Conf.

Samali, B. and Kwok, K. C. S., 1995, "Use of Viscoelastic Dampers in Reducing Wind- and Earthquake-induced Motion of Building Structures", Engineering Structures, Vol 17, No. 9, pp. 639-654.

Stoppa, M. H., 2003, "Modelagem Numérica e Projeto Ótimo de Sistemas de Amortecimento Viscoelásticos Aplicados a Placas Retangulares", Tese de Doutorado, Faculdade de Engenharia Mecânica, Universidade Federal de Uberlândia, Uberlândia-MG, Brasil.

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