A NEW TECHNIQUE TO IMPROVE COMPLETENESS CONDITIONS IN QUASI-DUAL BOUNDARY ELEMENT FORMULATION

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Abstract.*The Quasi-Dual formulation is a new technique to allow the application of Boundary Element approach to solve efficiently mathematical models associated of physical problems in which it is difficultr to obtain the inverse integral form. The advective-diffusive problems and non-homogeneous problems are some of the important problems in this last class. The current model uses a set of auxiliary independent functions and has difficulties to simulate two-dimensional problems with constant fluxes. The reason of this defficient behavior is probably due to absence of completeness. The present paper presents a first strategy to try to eliminate this problem, based on the introduction of new terms in the set of auxiliary functions. The aim is to improve completeness condition. As a result of proposed procedures, reciprocal matrices are generated at the final discretized equation system, in a similar way to Dual Reciprocity tecnhique. One typical example of constant flux is simulated with the new procedures and their results are discussed and analysed with details in this text.*

Keywords: *Diffusion, Convection, Boundary Element Method, Quasi-Dual Reciprocity, Numerical Simulation.*

1. Introduction

A remarkable advance to apply the Boundary Element Method in many important classes of problems was achieved by Nardini and Brebbia (1982) when they presented the Dual Reciprocity formulation, because with this formulation, free vibration problems, transient and dynamic problems and internal domain sources problems were successfully modeled by boundary elements. Partridge and Brebia (1992) presented a new and interesting generalization of the DRF to advective and non linear problems, but although it shows simplicity and flexibility, such model presents low accuracy level for many situations.

Recently another numerical procedure based in Dual Reciprocity idea was created: the Quasi-Dual formulation (Loeffler and Mansur, 2003). It has a similar structure of traditional Dual Reciprocity, specially by use of interpolation functions, but it is aimed specially to solve problems in which the mathematical model include the gradient operator in the basic variable. This is the case of advective and non-homogeneous problems, for example.

This different treatment of gradient operator term to improve the accuracy is very sensible. Traditional Dual Reciprocity formulation uses twice the interpolation domain functions in order to eliminate the gradient derivatives from the final equations. This procedure is responsible for accuracy loss. The approximation technique used by the Quasi-Dual Reciprocity produces substantial improvements in accuracy, because only once the gradient derivatives is approximated through the traditional domain interpolation procedure. It is important to point out that poles were not required by the Quasi-Dual approach, as well as the restriction of contant velocity field in advective cases does not exist.

For the cases in which the Peclet number is small, in other words, the diffusive phenomenon has still importance in the heat transfer process, the Quasi-Dual formulation have presented the best results. There are a lot of important applications, such as dispersion of effluents, whose physical situation fits in lower and moderate Peclet number. These are still quite unexplored in the Boundary Element Formulation. A lot of work will be developed in this area and similar fields, such as non-homogeneous problems.

Some papers were published presenting the good results of Quasi-Dual Reciprocity formulation. Initially it was applied to diffusive-advective problems: Loeffler and Mansur (2003) (already mentioned) and Massaro and Loeffler (1999). Preliminar reasonable results in Non-homogeneous problems can be noticed in Loeffler and Pereira (2004).

There is a slight loss of accuracy of the Quasi-Dual Reciprocity results in problems with uniform diffusive flux, though accuracy still remains better than that of the traditional Dual Reciprocity approach. In this sittuation better results must be obtained with the use a more complete set of interpolation functions, as described in this paper.

2. Quasi-Dual Formulation

The Quasi-Dual procedure has the same aim of original Dual Reciprocity Formulation. However, it uses a suitable approximation to model the convective term without necessity of new interpolation. The method can be applied in any mathematical models that contain gradient derivatives, but in this paper the procedure is ilustrated using the diffusive-advective equation, by convenience, because there are many interesting examples in this physical field.

So, let a control volume Ω represent a region in two-dimensional space where stationary potential fluid flow occurs. In indicial notation, the governing equation of advective-diffusive problems is given by:

$$K\theta,_{ii} - (v_i\theta),_i = 0 \tag{1}$$

The new form is regarding the incompressibility condition (Shames, 1973) in the potential fluid flow. This condition is basic for the use of Quasi-Dual Formulation. This allows to transform the convective term in two new terms, where just one is approximated by Dual Reciprocity interpolation procedure. The other receives the usual treatment of integral terms with BEM. Thus, the error of numerical model is very reduced. Using the traditional approach of BEM, equation (1) is put in strong integral form. The fundamental solution is also the diffusive one θ^*. So, one has:

$$K\int_\Omega \theta,_{ii}\,\theta^* d\Omega \;=\; \int_\Omega (v_i\theta),_i\,\theta^* d\Omega \tag{2}$$

The mathematical treatment of left hand side of equation (2) is trivial. By using integration by parts, the Divergence Theorem and the Dirac Delta's properties, it becomes:

$$K\int_\Omega \theta,_{ii}\,\theta^* d\Omega \;=\; K\left\{c(\xi)\theta(\xi) + \int_\Gamma \left[\theta q^* - q\theta^*\right] d\Gamma\right\} \tag{3}$$

The attention must be given to right hand side of mentioned equation. Using the integration by parts procedure and the Divergence Theorem, it can be writen:

$$\int_\Omega (v_i\theta),_i\,\theta^* d\Omega \;=\; \int_\Gamma v_i n_i \theta\theta^* d\Gamma \;-\; \int_\Omega v_i\theta\theta^*,_i\, d\Omega \tag{4}$$

The Quasi-Dual procedure approximates the second integral of left hand of equation (3) in order to transform it in a boundary integral. For this it is necessary to do:

$$b_i = v_i\theta \approx \alpha_p^j \Psi_{p,i}^j = \alpha_p^j \eta_{pi}^j \tag{5}$$

The diadic form of functions Ψ e η, more complex than usual vector form in scalar problems, is due to operational requests. Details for that will be presented later. Using the equation (5), it has:

$$\int_\Omega v_i\theta\theta^*,_i\, d\Omega = \alpha_p^j \int_\Omega \Psi_{p,i}^j \theta^*,_i\, d\Omega \tag{6}$$

Once more, using integration by parts and the Divergence Theorem:

$$\alpha_p^j \int_\Omega \Psi_{p,i}^j \theta^*,_i\, d\Omega = \alpha_p^j \int_\Gamma \Psi_p^j \theta^*,_i\, n_i d\Gamma - \alpha_p^j \int_\Omega \Psi_p^j \theta^*,_{ii}\, d\Omega \tag{7}$$

Using the diffusive fundamental solutions (Brebbia et al, 1982) and the Dirac Delta function properties, it is possible to write:

$$\alpha_p^j \int_\Omega \Psi_p^j \theta^*,_{ii}\, d\Omega = -\alpha_p^j c(\xi)\Psi_p^j(\xi) \tag{8}$$

Using the identity given by equation (3), the complete integral expression is given by:

$$K\left\{c(\xi)\theta(\xi) + \int_\Gamma \left[\theta q^* - q\theta^*\right] d\Gamma\right\} = -\int_\Gamma v_i n_i \theta\theta^* d\Gamma + \alpha_p^j\left[\int_\Gamma \Psi_p^j\theta^*,_i\, n_i d\Gamma + c(\xi)\Psi_p^j(\xi)\right] \tag{9}$$

In the last equation, q and q^* mean respectively the normal derivative of temperature and normal derivative of fundamental solution. The discretization procedure is the next stage. Using the well-known collocation method, the integrals can be rewritten in a matricial form given by:

$$H\theta - GQ = -B\theta + H\Psi\alpha \tag{10}$$

It is important to highlight that in the construction of B matrix it is possible to take acount of the variation of the velocity v_i along the boundary elements such as the distribution of the temperature and fluxes are made, that is, according to the order of shape function used. The following stage is to eliminate the α vector in equation (10).

$$\alpha = n^{-1}b = n^{-1}[v\theta] \tag{11}$$

The governing equation is a scalar one, but the source term taken lonely is vectorial. So, it is necessary to put the interpolation function η in the diadic form shown in equation (5). It is somewhat similar to the structure of function used in elastodynamic Dual Reciprocity procedure. The diadic type allows the inversion of matrix η, since suitable functions are chosen, avoiding singularities. One such a class of functions is given by:

$$\eta_{pi}^{j} = 3RR_iR_p + R^3\delta_{ip} \tag{12}$$

In the last expression, $R = R(Xj;X)$ is the Euclidian distance between the interpolation point X_j and the field point X, δ_{ij} is the Kronecker delta operator and:

$$R_p = \left[x_p\left(X_j\right) - x_p\left(X\right)\right] \tag{13}$$

Considering the equation (5), it is easily demonstrated that:

$$\Psi_p^j = R^3R_p \tag{14}$$

Finally, the complete matrix equation can be written as:

$$\left[H + B - M\right]\theta = GQ \tag{15}$$

3. Interpolation Functions Wideness

As mentioned earlier, the performance of Quasi-Dual Reciprocity formulation was superior to traditional Dual Reciprocity in diffusive-advective problems. Naturally, these good results can not be presented here due to space limitations. However, numerical investigations show that some problems require more complete interpolation functions to represent suitably the source term expressed by equation (5). This is the case of constant flux problems, for example. To solve these special cases with good accuracy, preliminar researches showed to be necessary to consider a new set of interpolation functions ρ_{pi}^j, in addition η_{pi}^j functions, to as shown next:

$$b_i \cong \alpha_p^j\left(\eta_{pi}^j + \rho_{pi}^j\right) \tag{16}$$

The additional domain integral that appear due to ρ_{pi}^j can be transformed as indicated below

$$\alpha_p^j\int_\Omega \rho_{pi}^j\theta^*{,_i}\, d\Omega = \alpha_p^j\int_\Omega\left(\rho_{pi}^j\theta^*\right){,_i}\, d\Omega - \alpha_p^j\int_\Omega \rho_{pi,i}^j\theta^* d\Omega \tag{17}$$

The main feature of the dyadic function ρ_{pi}^j is that its divergent is null, that is:

$$\rho_{pi,i}^j = 0 \tag{18}$$

Due to this condition it is possible to express the left-hand-side of equation (17) to an unique boundary integral, that is, the second term on the right-hand-side of equation (16) is null, and the first term can be easily transformed into a boundary integral by direct use of the divergence theorem, as indicated next:

$$\alpha_p^j\int_\Omega\left(\rho_{pi}^j\theta^*\right){,_i}\, d\Omega = \alpha_p^j\int_\Gamma \rho_{pi}^j n_i\theta^* d\Gamma \tag{19}$$

There are some possible choices of ρ_{pi}^j, one of them is:

$$\rho_{pi}^j = \Delta\left(3RR_iR_p - 4R^3\delta_{ip}\right) \tag{20}$$

In former equation Δ is a constant. Now, the complete matrix system is:

$$H\Theta - GQ = -B\Theta + \left(H\Psi + G\Lambda\right)\alpha \tag{21}$$

Where the Λ matrix is related to the new functions ρ_{pi}^j , according to the following form:

$$\lambda_p^j = \rho_{pi}^j n_i \tag{22}$$

Operating vector α in a similar way already shown, that is:

$$\alpha = \left(\eta + \rho\right)^{-1} b \tag{23}$$

The complete matrix system can be written as:

$$H\theta - GQ = \left(-B + T\right)\theta \tag{24}$$

4. Example: Circumferencial fluid flow with radial temperature gradient

The example chosen to ilustrate the effect of a more complete set of interpolation functions consists of a pseudo-convective case. A fluid flows through two circular boundaries as shown in the figure 1a. A difference of temperatures is prescribed between the two curved faces. Physically this problem might be understood as that of two pipe sections isolated internally by a fluid. Symmetry properties can be used to reduce the BEM mesh required to model this case, as shown in the figure 1b. For the sake of simplicity, the rotational velocity of the fluid flow can be regulated by prescribing an angular velocity ω, represented here by Peclet number. Other physical and geometrical features are also presented in figure 1b.

The heat transfer occurs only by diffusion, but the advective phenomenon is numerically computed by the mathematical formulation and it should have null effect, except for numerical inaccuracy. So, the radial behavior of the temperature is governed simply by the following equation:

$$\theta = \frac{T_2 - T_1}{ln\,(b/a)}[ln(r) - ln(a)] + T_1 \qquad (25)$$

The radial derivative of temperature is given by:

$$\frac{d\theta}{dr} = \frac{T_2 - T_1}{ln\,(b/a)}\left(\frac{1}{r}\right) \qquad (26)$$

In this example, to ilustrate better the behavior of more complete interpolation functions, the results of Quasi-Dual Reciprocity formulation are compared with traditional Dual Reciprocity results. Both formulations used 39 constant boundary elements of equal lenght in this analysis. However, for traditional Dual Reciprocity formulation 24 poles were required. Poles were not used for the Quasi-Dual Reciprocity procedure.

To solve suitably problems with this features, that is, problems in which exist strong constant diffusive fluxes on the boundary, it is necessary to put into the formulation an interpolation function like that given by equation (15). Figure (2) shows the percentage error of temperatures along the radial direction as a function of Peclet number, and figure (3) displays the percentage average error for fluxes over the curved internal and external boundaries also a function of Peclet number. Both Quasi-Dual and traditional Dual results are presented.

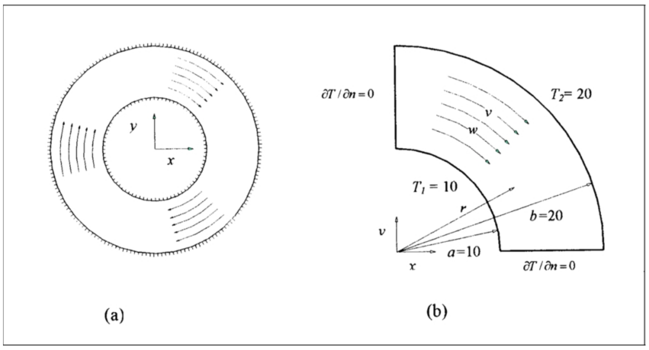

Figure 1. Geometric and physical features of fourth example.

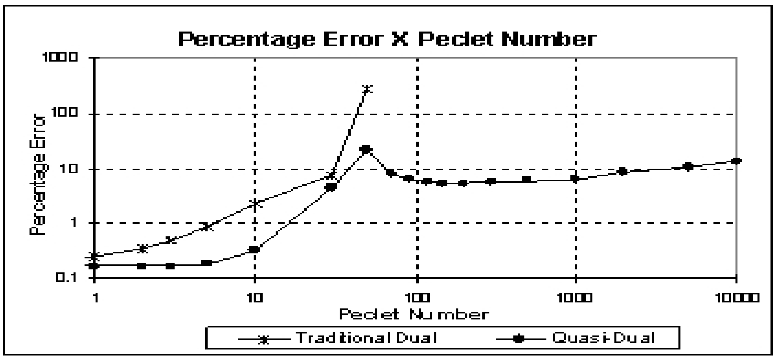

Figure 2. Percentage average error of temperature along radial direction.

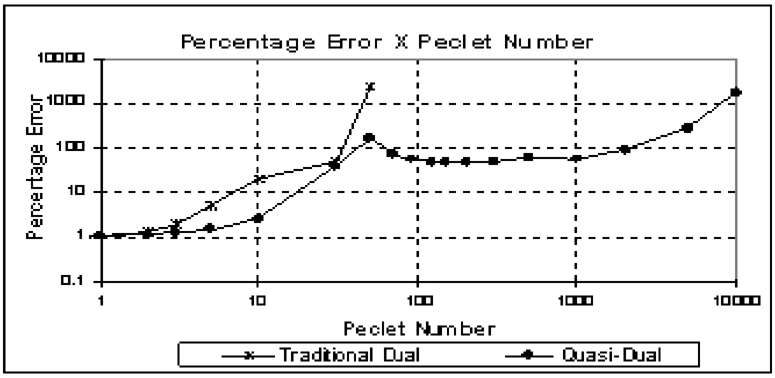

Figure 3. Percentage error of fluxes over internal and external curved faces.

In this example results for fluxes of both Quasi-Dual and Dual Reciprocity formulations deteriorate as Peclet number becomes large, however Quasi-Dual temperature and fluxes results are better. Results may improve if errors due to representation of circular boundaries by straight elements are removed either by refining the mesh of by using higher order elements for the geometry.

In the following figures it is shown the behavior of Quasi-Dual results for temperature and diffusive flux with the mesh refinement. Peclet number was assumed to be equal to 10. Meshes with 13, 39, 52 and 107 boundary elements were used in this numerical experience. Figures (4) and (5) show respectively that the temperature and diffusive flux errors decrease as the number of boundary elements increases.

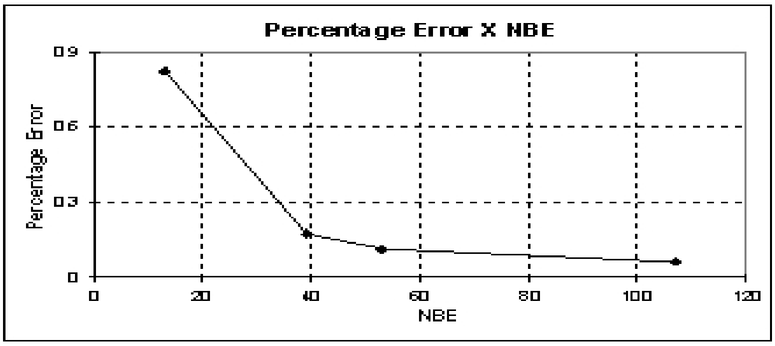

Figure 4. Percentage error of temperature values along radial direction. Peclet number is 10

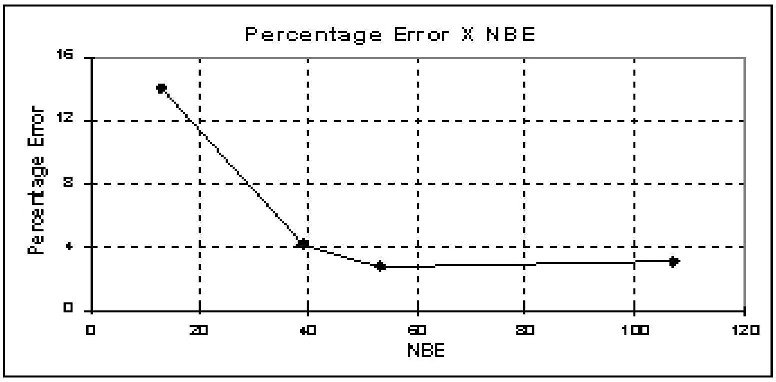

Figure 5. Percentage error of diffusive flux in internal and external boundaries. Peclet number is 10

In these last tests it can be noted that the refinement was effective to reduce the error of temperature values, but the same did not occur to diffusive flux, whose error remained around 3%, despite mesh refinement. Errors obtained with standard Dual Reciprocity analysis for Peclet 10 for this example were too high and therefore were not presented here.

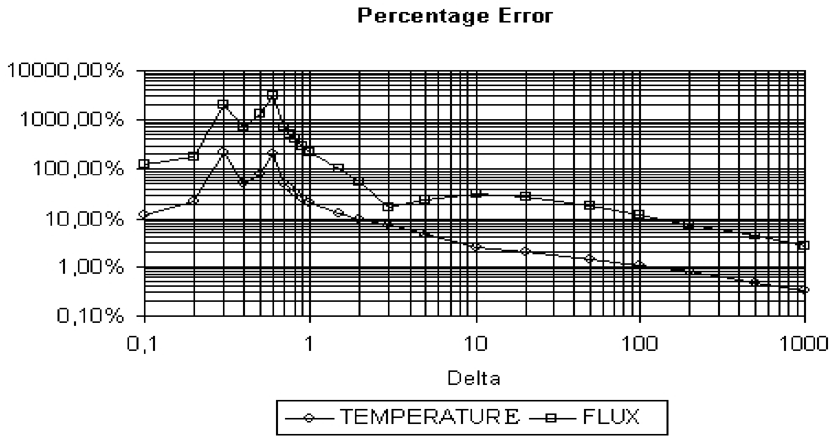

Figure 6. Percentage error as a funciton of delta constant.

Despite of the reasonable results acchieved by the present technique, threre are some details that need to be understood. The constant Δ shown in equation (20) has influence in the quality of results. It seems to be a function of Peclet number. High values of Δ imply in a predominance of advective features of interpolation functions. This behavior can be observed in figure (6). It must be related that in all the simulations presented here, was employed $\Delta = 3$. Until the moment it is not possible to stablish an accurate and closed formula to relate directly Δ with the advective contents of the problem, because the boundary element size also has an important influence.

5. Conclusions

The Quasi-Dual Reciprocity is a new and unexplored Boundary Element Formulation. Tests and adaptations will be implemented to give the suitable reliability for larger employment of this formulation.

For the cases in which the Peclet number is small, in other words, the diffusive phenomenon has still importance in the heat transfer process, the Quasi-Dual formulation have presented the best results. There are a lot of important applications, such as dispersion of effluents, whose physical situation fits in lower and moderate Peclet number.

However, it was detected a slight loss of accuracy of the Quasi-Dual Reciprocity results in Problems with uniform diffusive flux, though accuracy still remains better than that of the traditional Dual-Reciprocity approach. In this particular situation better results can be obtained if one uses a more complete set of interpolation functions as exposed in the text. However, the completely satisfactory accuracy still does cannot be reached. The indetermination about the value of delta coefficient is a important problem to be solved. No doubt that the better definition about delta coefficient is the next research topic to be investigated concerning the Quasi-Dual Reciprocity formulation.

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