NON-LINEAR ANALYSIS OF BUILDING FLOOR STRUCTURES BY THE BOUNDARY ELEMENT METHOD

Gabriela R. Fernandes
Unesp - São Paulo State University, Al. Bahia, 550, 15385-000 Ilha Solteira, Brazil
e-mail:grezfernandes@uol.com.br

Wilson S. Venturini
São Carlos School of Engineering, University of São Paulo, Av. Trabalhador São Carlense 400, 13566-970 São Carlos, Brazil
e-mail: venturin@sc.usp.br.

Abstract *In this work, a formulation of the boundary element method (BEM) to perform non-linear analysis of building floor structures is proposed. The building floor is considered as a zoned domain plate, where each sub-domain represents a slab or a beam. The model is based on the Kirchhoff's hypothesis and the membrane effects are taken into account in the analysis. In order to reduce the number of degrees of freedom, the kinematics Navier-Bernoulli hypothesis is assumed to simplify the strain field for the thin sub-regions (beams). The non-linear formulation is obtained by incorporating initials stress fields into the linear formulation. Those initial stresses allow the determination of the corrector vector that must be considered in the non-linear analysis. The domain integrals involving those stresses are performed by using the well-known cell sub-division. In the incremental-iterative procedure is used the consistent tangent operator, while Von Mises criterion is adopted to govern the non-linear behaviour. The moments and normal forces are computed from the stress numerical integration using a Gaussian scheme the plate thickness. Finally, numerical examples are analysed to show the result accuracy.*

Keywords: *boundary elements, plate bending, building floor structures, non-linear analysis*

1. Introduction

The direct BEM formulation applied to Kirchhoff's plates has appeared in the seventies (Bézine, 1978; Stern, 1979; Tottenhan, 1979). These works, as well as several other more recent publications, have demonstrated that the method is a robust numerical technique to deal with plates in bending, taking into account its accuracy and confidence. Among those works, it is important to mention the papers written by Song (1986), Hartmann and Zotemantel (1986), Oliveira Neto and Paiva (1998).

Using BEM coupled with finite elements is the natural numerical procedure to analyze building floor structures, where plate and beam elements are modeled, respectively, by boundary elements and finite elements. However, for complex floor structures the number of degrees of freedom increases rapidly and the accuracy of the solution diminishes. An alternative scheme to deal with zoned domains without dividing them into sub-regions has been proposed by the author (Venturini and Paiva, 1993; Venturini, 1992). In this formulation, only displacements are defined along the interfaces. This formulation has been successfully extended to analyse bending effects of plates reinforced by beam elements. The beam elements were treated as thin sub-regions for which some convenient cinematic approximations were assumed (Fernandes and Venturini, 2002). Then, this formulation has been modified to deal with the general case of plates reinforced by beams, in which the plate and beam elements are not necessarily defined in the same plane. Therefore, the resulting integral equations of both, the bending and the plane stress problems are coupled and cannot be treated separately. All displacement components across the beam width are assumed linear therefore reducing significantly the number of internal unknowns (Fernandes and Venturini, 2005).

In this paper, the linear formulation described above is now extended in order to perform non-linear analysis of stiffened plates bending taking into account the membrane effects. Some numerical examples are then presented to illustrate the accuracy of the results.

2. Basic Equations

Without loss of generality, let us consider the three sub-region plate depicted in Fig. 1 which represents a particular floor structure composed by two slabs, denoted by the sub-domains Ω_1, Ω_2, , and one beam, defined by the sub-domain Ω_3. All sub-regions are referred to a Cartesian system of co-ordinates with x_1 and x_2 axes laying on a chosen reference surface In figure 1 t_1, t_2 and t_3 are the sub-domains thicknesses; c_1, c_2 and c_3 define the distances of the corresponding middle surface to the reference one; Γ_j is the external boundary related to the sub-region Ω_j; Γ gives the total external boundary and Γ_{jk} specifies the interface where the subscript indicates the adjacent sub-regions.

The non-linear formulation is obtained by assuming that the plate is subjected to initials fields of moment and membrane forces. In a linear analysis those initials efforts could be due to temperature changers, for instance. In a non-linear analysis they are the plastic efforts fields that must be considered in the non-linear process in order to obtain the plate equilibrium. In this case, the total strain tensor has two components: the elastic one (ε_{ij}^{e}) due to the loads acting on the plate surface and the plastic one (ε_{ij}^{p}) related to the initials efforts. On the other hand, the formulation of bending plate problem taking into account the membrane effects is obtained by coupling the bending and stretching problems. Thus, the resulting total strain tensor is given by:

$$\varepsilon_{ij} = \left(\varepsilon_{ij}^{e(S)} + \varepsilon_{ij}^{e(B)}\right) + \left(\varepsilon_{ij}^{p(S)} + \varepsilon_{ij}^{p(B)}\right) \tag{1}$$

where the superscripts S and B, are referred, respectivaly, to stretching and bending problem.

Adopting the Kirchhoff´s hypothesis the bending component is given by: $\varepsilon_{ij}^{B} = -x_3 w,_{ij}$ with $w,_{ij}$ being the plate surface curvature.

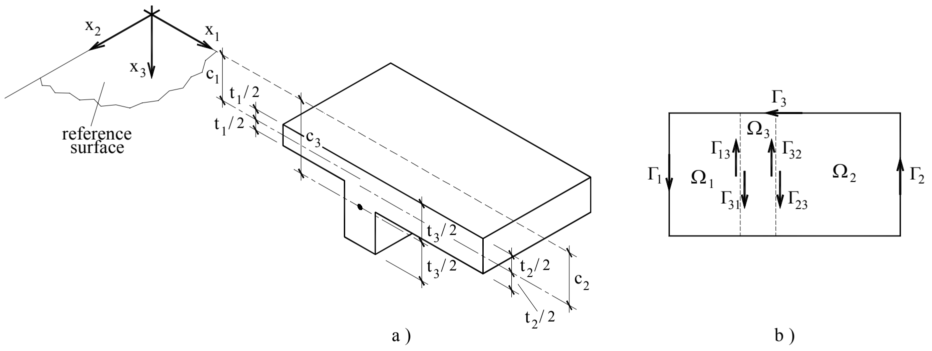

Figure 1. a) General zoned plate domain; b) reference surface view

By applying the generalized Hooke´s law one obtains the stress components which lead to the following internal forces after performing the integrals along the plate thickness.

$$m_{ij} = m_{ij}^{e} - m_{ij}^{p} \tag{2}$$

$$N_{ij} = N_{ij}^{e} - N_{ij}^{p} \tag{3}$$

where m_{ij} and N_{ij} are the internal forces due to the loading; the efforts with subscripts e and p are, respectively, the elastic and plastic components, being the elastic efforts given by:

$$m_{ij}^{e} = -D\left[\nu w,_{kk} \delta_{ij} + (1-\nu) w,_{ij}\right] \tag{4}$$

$$N_{ij}^{e} = \frac{\overline{E}}{\left(1-\nu^2\right)}\left[\nu \varepsilon_{kk}^{S} \delta_{ij} + (1-\nu)\varepsilon_{ij}^{S}\right] \tag{5}$$

in which $\overline{E} = Et$, $D = Et^3/(1-\nu^2)$ is the flexural rigidity, with E being the Young's modulus and ν the Poisson's ration.

Assuming, initially, that the plate supports only distributed load g acting on the sub-regions middle plane, in the x_3 direction, for a point placed at any of the sub-domains one can write the following equilibrium equations:

$$q_j = -Dw,_{kkj} - m_{ij,j}^{p} \tag{6}$$

$$w,_{kkll} = \frac{\left(g - m_{ij,ij}^{p}\right)}{D} \tag{7}$$

where q_j represents shear forces.

As the Eq. (7) is a fourth order equation, in the formulation there must be only four boundary values independents. In order to eliminate one boundary value, Kirchhoff in 1850 demonstrated that the twisting moments and shear forces can be combined together leading to the effective shear force V_n:

$$V_n = q_n + \frac{\partial m_{ns}}{\partial s} \tag{8}$$

where *(n,s)* are the local co-ordinate system, with n and s referring to the boundary normal and tangential directions, respectively.

Let us now consider only the stretching problem with loads acting in the x_1 and x_2 directions. In this case the in-plane equilibrium gives the following equation:

$$u_{i},_{jj} + \frac{1}{1-2\nu'} u_{j},_{ij} + \frac{b_i}{\overline{G}} - \frac{N^p_{ij,j}}{\overline{G}} = 0 \tag{9}$$

where b_i represents the corresponding body forces distributed over the plate middle surface; $\overline{G} = Gt$ and $\nu' = \nu/(1+\nu)$ (for plane stress conditions that is the case considered in this work), being G the shear elastic modulus.

3. Integral Representations for Zoned Domain Plates

The non-linear solution for zoned domain plates, taking into account the membrane effects, is obtained from Betti's reciprocal theorem, given in terms of two stress and deformations states: one referred to the actual plate problem and the other one related to the well-known fundamental problem. Let us consider initially, all variables referred to the sub-region middle surface. Dividing the stress and deformations into the stretching and bending components and performing the integrals along the thickness, one obtains the reciprocity relation written in terms of elastic efforts. Considering Eqs. (2) and (3), and treating separately the stretching and bending problems, for a particular sub-region m, the Betti's theorem can be written as follows:

$$\int_{\Omega_m} \varepsilon^{Sm*}_{ijk} N^m_{jk} d\Omega = \int_{\Omega_m} N^{m*}_{ijk} \varepsilon^{Sm}_{jk} d\Omega - \int_{\Omega_m} \varepsilon^{Sm*}_{ijk} N^{m(p)}_{jk} d\Omega \tag{10a}$$

$$\int_{\Omega_m} w,^{m*}_{jk} m^m_{jk} d\Omega = \int_{\Omega_m} m^{m*}_{jk} w,^m_{jk} \, d\Omega - \int_{\Omega_m} w,^{m*}_{jk} m^{m(p)}_{jk} d\Omega \tag{10b}$$

All fundamental values (variables with superscript *) used in this work are omitted; they are given in the specialized literature (e.g. Brebbia et al., 1997). For convenience the fundamental values of deformations will be written in terms of those related to the sub-region where is placed the source point $\boldsymbol{q}$, as follows:

$$\varepsilon^{Sm*}_{ijk} = \left[Et/(E_m t_m)\right] \varepsilon^*_{ijk} \tag{11.a}$$

$$w,^{m*}_{jk} = \left[D/D_m\right] w,^*_{jk} \tag{11.b}$$

where t_m and D_m are thickness and plate rigidity of sub-region Ω_m, while D and t are the corresponding reference values of the sub-region where the load point was adopted.

In expressions (10) the deformations ε^{Sm}_{jk} and the moments m^m_{jk} are referred to the sub-region middle surface. Writing these values in terms of those on the reference surface:

$$\varepsilon^{Sm}_{jk} = \varepsilon_{jk} - c_m w,_{jk} \tag{12a}$$

$$m^m_{jk} = m_{jk} - c_m N_{jk} \tag{12b}$$

where ε_{jk} and m_{jk} are the strain and moment values referred to the reference surface (note that the normal force and the curvature don't change along the thickness).

Replacing equations (11) and (12) into Eq. (10) and aplying the resulting equations for all sub-regions, one obtains the stretching and bending reciprocity relations for the whole plate:

$$\int_{\Omega} \varepsilon^*_{ijk} N_{jk} d\Omega = \sum_{m=1}^{N_{sub}} \frac{E_m t_m}{Et} \left[\int_{\Omega_m} N^*_{ijk} \varepsilon_{jk} d\Omega - c_m \int_{\Omega_m} N^*_{ijk} w,_{jk} d\Omega \right] - \int_{\Omega} \varepsilon^*_{ijk} N^p_{jk} d\Omega \tag{13a}$$

$$\int_{\Omega} w,^*_{jk} \, m_{jk} d\Omega = \sum_{m=1}^{N_{sub}} \left[\frac{D_m}{D} \int_{\Omega_m} w,_{jk} \, m^*_{jk} d\Omega + c_m \int_{\Omega_m} w,^*_{jk} \left(N_{jk} + N^p_{jk}\right) d\Omega \right] - \int_{\Omega} w,^*_{jk} m^p_{jk} d\Omega \tag{13b}$$

in which N_{sub} is the sub-regions number.

Equations (13) can be integrated by parts to give the integral representations of displacements. For the stretching problem one obtains:

$$- K_{w,_i} w,_i + K_{u_i} u_i = -\sum_{m=1}^{N_{sub}} \frac{E_m t_m}{Et} \int_{\Gamma_m} \left(u_n p_{in}^* + u_s p_{is}^* \right) d\Gamma + \sum_{m=1}^{N_{Sub}} \frac{E_m t_m}{Et} c_m \int_{\Gamma_m} \left(p_{in}^* w,_n + p_{is}^* w,_s \right) d\Gamma +$$

$$- \sum_{j=1}^{N_{\text{int}}} \frac{\left(E_b t_b - E_a t_a\right)}{Et} \int_{\Gamma_{ba}} \left(u_n p_{in}^* + u_s p_{is}^* \right) d\Gamma + \sum_{j=1}^{N_{\text{int}}} \frac{\left(E_b t_b c_b - E_a t_a c_a\right)}{Et} \int_{\Gamma_{ba}} \left(p_{in}^* w,_n + p_{is}^* w,_s \right) d\Gamma +$$

$$+ \int_{\Gamma} \left(u_{in}^* p_n + u_{is}^* p_s \right) d\Gamma + \int_{\Omega_b} \left(u_{in}^* b_n + u_{is}^* b_s \right) d\Omega + \int_{\Omega} \varepsilon_{ijk}^* N_{jk}^p d\Omega \tag{14}$$

where, Γ_m is the external boundary of sub-region Ω_m, Γ_{ba} represents a beam interface, being the subscripts ***b*** and ***a*** referred, respectively, to the beam sub-region and to the corresponding adjacent sub-region, N_{int} is the number of beam interfaces; n and s are the local normal and shear direction co-ordinates; for an internal point $K_{w,i}=c_R$ and $K_{ui}=1$; for boundary points $K_{w,i}=c_R/2$ and $K_{ui}=1/2$; for interface points $K_{wi} = \frac{1}{2}\left(c_R + \frac{E_a t_a c_a}{Et} \right)$ and $K_{ui} = \frac{1}{2}\left(1 + \frac{E_a t_a}{Et} \right)$, where c_R is the distance of the sub-region middle surface where the collocation point is placed to the reference one.

For the bending problem, the integral representation of deflections is:

$$K(q)w(q) = -\sum_{m=1}^{N_{Sub}} \frac{D_m}{D} \int_{\Gamma_m} \left(V_n^* w - M_n^* \frac{\partial w}{\partial n} \right) d\Gamma - \sum_{j=1}^{N_{c1}} \frac{D_j}{D} R_{cj}^* w_{cj} - \sum_{j=1}^{N_{c2}+N_{c3}} \left(\frac{D_j - D_a}{D} \right) R_{cj}^* w_{cj} +$$

$$- \sum_{j=1}^{N_{\text{int}}} \int_{\Gamma_{ba}} \left(\frac{D_b - D_a}{D} \right) \left(V_n^* w - M_n^* \frac{\partial w}{\partial n} \right) d\Gamma + \sum_{j=1}^{N_{c1}} R_{cj} w_{cj}^* + \int_{\Gamma} \left(V_n w^* - M_n \frac{\partial w^*}{\partial n} \right) d\Gamma +$$

$$+ \sum_{m=1}^{N_{sub}} c_m \int_{\Gamma_m} \left[p_n w,_n^* + p_s w,_s^* \right] d\Gamma + \sum_{j=1}^{N_{\text{int}}} \left(c_b - c_a \right) \int_{\Gamma_{ba}} \left[p_n w,_n^* + p_s w,_s^* \right] d\Gamma - \int_{\Omega} w,_{jk}^* m_{jk}^p d\Omega +$$

$$- \sum_{m=1}^{N_{sub}} c_m \int_{\Omega_m} \left(b_n w,_n^* + b_s w,_s^* \right) d\Omega + \int_{\Omega_g} g w^* d\Omega + \sum_{m=1}^{N_{sub}} c_m \int_{\Omega_m} w,_{jk}^* N_{jk}^p d\Omega \tag{15}$$

where N_{c1}, N_{c2} and N_{c3} are, respectively, numbers of corners between boundary elements, between interface and boundary elements and between interface elements (see Fernandes and Venturini, 2002); Ω_g is the plate loaded area and the free term is given by: $K(q)=1$ for internal point; $K(Q)=0.5$ for a boundary point; $K(Q)=0.5(1+D_a/D)$ for an interface point; the free terms related to the corners are defined in Fernandes and Venturini (2002).

In eq. (14) and (15) all variables are referred to the reference surface and he rotation $w,_s$ (both on the interfaces and on the boundary) has been eliminated by replacing its value by numerical derivatives of w. Note that the integral representations of the rotations ($w,_n$, $u_{n,n}$ or $u_{s,n}$) can be easily obtained by differentiating equations (14) or (15).

Let us now consider the beam B_3 represented in Fig. (2) by the sub-region Ω_3. In order to reduce the number of degrees of freedom related to the interfaces, some Kinematic hypothesis will be assumed along the beams cross sections. The interface displacement vector related to the beam interfaces are translated to the skeleton line, as follows:

$$u_k^{\Gamma_{32}} = u_k + u_k,_n b_3 / 2 \qquad \text{k=n,s} \tag{16a}$$

$$u_k^{\Gamma_{31}} = -\left[u_k - u_k,_n b_3 / 2 \right] \tag{16b}$$

$$w^{\Gamma_{32}} = w + w,_n b_3 / 2 \tag{16c}$$

$$w^{\Gamma_{31}} = w - w,_n b_3 / 2 \tag{16d}$$

where $u_k^{\Gamma_{ij}}$ and $w^{\Gamma_{ij}}$ are related to the interfaces Γ_{31} and Γ_{32}; u_k, w, $u_k,_n$ and $w,_n$ are referred to the skeleton line.

Assuming that the stress field is constant along the direction perpendicular to the beam axis from equation (5) for the plane stress case, the tractions along the interfaces Γ_{31} and Γ_{32} are given by:

$$p_k^{\Gamma_{32}} = -p_k^{\Gamma_{31}} = p_k^c = \overline{G}\left[\frac{2\nu}{(1-\nu)} u_{\ell},_{\ell} n_k + (u_k,_n + u_n,_k) \right] \qquad k=s,n \tag{17}$$

where n and s are defined along the beam axis; the derivatives with respect to direction s, ($u_{n},_s$ and $u_{s},_s$) are replaced by numerical differences.

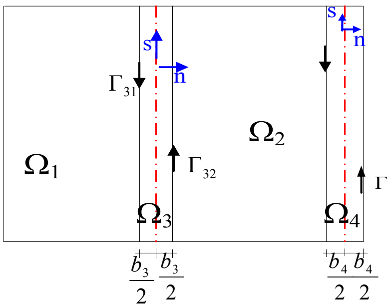

Figure 2. reinforced plate view

In this model, as the variables are referred to the skeleton line instead of the interfaces, the number of equations required has been strongly reduced. Note that in the case of external beams, only the interface tractions are approximated.

The elastic moments are computed from Eq. (4), in which the integral representations for curvatures $w_{,lk}$ can be written by differentiating Eq. (15) twice. The elastic membrane forces are computed from Eq. (5), where the deformations can be written in terms of the displacements derivatives whose integrals are obtained by differentiating Eq. (14). It is important noting that in the curvature and displacements derivatives representations the domain integrals involving the plastic efforts present singularity of order $1/r^2$, requiring therefore proper care to be evaluated.

4 BEM algebraic equations

Let us consider the case presented in the previous section where the beam variables are defined on their axes and the tractions p_n and p_s are approximated according to the Eq. (17). The integral representations derived in section (3) can be transformed into algebraic equations by discretizing the boundary and beam axes into elements and the plate domain into cells, where the problem variables are approximated. In this work were adopted linear elements both on the boundary and beam axis in which quadratic shape functions are used to approximate the respective variables (w, u_n, u_s $w_{,n}$, $u_{n,n}$, $u_{s,n}$ V_n, M_n p_n and p_s). For the domain discretization were considered three different types of cells: triangular cells in the slabs with linear approximation for the initial efforts; rectangular cells in the beams with constant approximation along the cross section; one rectangular cell at beams intersections with constant approximation for the variables.

On the boundary are defined the following nodal values: the generalized displacements: w, u_n, u_s and $w_{,n}$, as well as the generalized forces: V_n, M_n p_n and p_s. Therefore, for each boundary node are required four equations. It has been adopted one deflection equation (Eq. 15) and two in-plane displacement relations (Eq. 14) written for a point placed on the boundary and one deflection equation (Eq. 15) written for an external point very near the boundary.

The skeleton nodal values are the generalized displacements w, u_n, u_s, $w_{,n}$, $u_{n,n}$ $u_{s,n}$. Their counterpart values were all eliminated. As all this variables remain as unknowns, we have to write six algebraic relations. All of the adopted equations are written for collocations defined along the skeleton line: two in-plane displacement relations (Eq. 14), one deflection relation (Eq. 15), two in-plane displacement derivative relations and one slope relation.

For both, boundary and interface corners, extra equations must be written if the reactions are preserved as unknowns.

After discretizing the plate domain into cells, six extra equations are required for each cell node: three elastic moment equations (Eq. 4) plus three elastic normal forces equations (Eq. 5). Selecting the recommended collocation points and writing the corresponding algebraic relations for all of them, one obtains the following set of equations given in terms of boundary and beam axis values, as well as the initial efforts:

$$\boldsymbol{HU} = \boldsymbol{GP} + \boldsymbol{T} + \boldsymbol{E}_M \boldsymbol{M}^P + \boldsymbol{E}_N \boldsymbol{N}^P \tag{18}$$

where $\{U\}$ and $\{P\}$ are, respectively, displacements and tractions vector; $\{T\}$ is the independent vector due to the applied loads; $\{M^P\}$ and $\{N^P\}$ contains plastic moments and normal forces at cells nodes; $[H]$ and $[G]$ are matrices achieved by integrating all boundary and interfaces, while the matrix $[E]$ is obtained by performing the integrals over the cells.

After writing the elastic moments and normal forces equations for all cells nodes, one obtains the followings algebraic equations:

$$\boldsymbol{M}^e = -\boldsymbol{H}'\boldsymbol{U} + \boldsymbol{G}'\boldsymbol{P} + \boldsymbol{T}' + \boldsymbol{E}'_M \boldsymbol{M}^P + \boldsymbol{E}'_N \boldsymbol{N}^P \tag{19}$$

$$\boldsymbol{N}^e = -\boldsymbol{H}''\boldsymbol{U} + \boldsymbol{G}''\boldsymbol{P} + \boldsymbol{T}'' + \boldsymbol{E}''_M \boldsymbol{M}^P + \boldsymbol{E}''_N \boldsymbol{N}^P \tag{20}$$

In non-linear formulation is convenient to arrange Eq. (18), (19) and (20) in order to express them in terms of linear solution and plastic efforts as follows:

$$\boldsymbol{X} = \boldsymbol{L} + \boldsymbol{R_M M^P} + \boldsymbol{R_N N^P} \tag{21}$$

$$\boldsymbol{M^e} = \boldsymbol{K} + \boldsymbol{S_M M^P} + \boldsymbol{S_N N^P} \tag{22}$$

$$\boldsymbol{N^e} = \boldsymbol{K'} + \boldsymbol{S'_M M^P} + \boldsymbol{S'_N N^P} \tag{23}$$

In Eq. (21) the vector X contains the problem unknowns on the boundary, beam axis and corners. The vectors L, K and K', given in Eq. (21), (22) and (23), are corresponding to the elastic solution due to the prescribed generalized loads acting along the boundary or over the domain, while M^p and N^p effects are represented by matrices R, S and S'.

5 Implicit BEM Formulation using the Consistent Tangent Operator

In order to obtain the consistent tangent operator let us consider Eq. (2) and (3) corresponding to the moments and normal forces in the plate, where the elastic generalized forces are given by expressions (22) and (23). From these equations the followings plate equilibrium relations can be written:

$$\{K\} + [S_M]([C_m]\{(1/r)\} - \{M\}) - [I][C_m]\{(1/r)\} + [S_N]([C_N]\{\varepsilon^s\} - \{N\}) = 0 \tag{24}$$

$$\{K'\} + [S'_N]([C_N]\{\varepsilon^s\} - \{N\}) - [I][C_N]\{\varepsilon^s\} + [S'_M]([C_M]\{(1/r)\} - \{M\}) = 0 \tag{25}$$

where $[C_N]$ e $[C_m]$ are elastics rigidity tensors obtained form Hooke's law; $\{1/r\}$ is the curvature vector; $\{\varepsilon^S\}$ is the in-plane deformations; $[I]$ the identity matrix; M and N the actual generalized forces.

Defining the vector $\{F^{ext}\}_{n+1}^{t} = \{M^{ext} \quad N^{ext}\}_{n+1}$ as the external moment and normal forces vector and $\{F\}_n^{i(t)} = \{M^{\text{int}} \quad N^{\text{int}}\}_n^i$ as the internal moment and normal forces vector, in an iteration i related to an increment n, the followings equilibrium relations must be verified:

$$\{M^{ext}\}_{n+1} - \{M^{\text{int}}\}_n^{i+1} - \{\Delta M^{\text{int}}\}_n^{i+1} = 0 \tag{26}$$

$$\{N^{ext}\}_{n+1} - \{N^{\text{int}}\}_n^{i+1} - \{\Delta N^{\text{int}}\}_n^{i+1} = 0 \tag{27}$$

By applying the Newton Raphson method, equations (26) and (27) can be written together leading to the following set of equations:

$$\{\Delta F^e\}_n^{i+1} = [K^{TC}]_n^i \, \delta\{\Delta\varepsilon\}_n^i \tag{28}$$

where $\{\Delta F^e\}_n^{i+1} = \{\Delta M^e \quad \Delta N^e\}_n^{i+1}$ is the vector for elastic generalized forces increments, $\delta\{\Delta\varepsilon\}_n^{i(t)} = \{\delta\{\Delta(1/r)\} \quad \delta\{\Delta\varepsilon^S\}\}_n^i$ is the corrector vector for curvatures and in-plane deformations; $[K^{TC}]_n^i$ is the tangent consistent operator given by: $[K^{TC}]_n^i = \begin{bmatrix} [K_M]_n^i & [\overline{K}_S]_n^i \\ [\overline{K}_M]_n^i & [K_S]_n^i \end{bmatrix}$ with its matrices defined as: $[K_M]_n^i = \dfrac{\partial\{\Delta M^{\text{int}}\}_n^i}{\partial\{\Delta(1/r)\}_n^i} = \left[[S_M]\left([C_m^{ep}]_n^i - [C_m]\right) + [I][C_m]\right]$, $[\overline{K}_S]_n^i = \dfrac{\partial\{\Delta M^{\text{int}}\}_n^i}{\partial\{\Delta(\varepsilon^S)\}_n^i} = [S_N]\left([C_N^{ep}]_n^i - [C_N]\right)$, $[\overline{K}_M]_n^i = \dfrac{\partial\{\Delta N^{\text{int}}\}_n^i}{\partial\{\Delta(1/r)\}_n^i} = \left[[S'_M]\left([C_m^{ep}]_n^i - [C_m]\right)\right]$, $[K_S]_n^i = \dfrac{\partial\{\Delta N^{\text{int}}\}_n^i}{\partial\{\Delta\varepsilon^S\}_n^i} = [S'_N]\left([C_N^{ep}]_n^i - [C_N]\right) + [I][C_N]$, where $[C_N^{ep}]_n^i = \int_{-t/2}^{t/2} [C^{ep}]_n^i dx_3$ and $[C_m^{ep}]_n^i = \int_{-t/2}^{t/2} (x_3)^2 [C^{ep}]_n^i dx_3$, being $[C^{ep}]_n^i$ the local tangent matrix, obtained from the adopted constitutive model.

Denoting β_n as the load factor, in Eq. (28) the elastic generalized forces increments are given by $\{\Delta M^e\}_n^{i+1} = \beta_n \{K\}$ and $\{\Delta N^e\}_n^{i+1} = \beta_n \{K'\}$ if *i=0*; if $i \geq 1$ they are the following ones: $\{\Delta M^e\}_n^{i+1} = \{\Delta M^e\}_n^i - \{\Delta M\}_n^i = \{M^p\}_n^i$ and $\{\Delta N^e\}_n^{i+1} = \{\Delta N^e\}_n^i - \{\Delta N\}_n^i = \{N^p\}_n^i$.

The plate thickness is divided into stations defined by a numerical Gaussian scheme. The stresses are computed at each Gauss point where the Von Mises criterion is verified. After checking the criterion at all stations, one obtains the actual stress distribution along the thickness and then, the actual moments and normal forces can be computed numerically by performing the stress integrals across the plate thickness. Thus, the generalized actual forces will be approximated according to the Gauss scheme adopted and the number of stations (N_g) taken to perform the integrals. After computing the plastic moment and normal forces increments, the convergence criterion is checked. If the convergence criterion is not verified for any of the cells nodes other iteration must be considered; otherwise the analysis goes to the next increment.

6. Numerical Examples

In what follows, it will be presented two numerical examples referred to a same stiffened plate. In the first test (Fig. 3) the stiffened plate will be subjected to simple bending, therefore $c_B=0cm$ for the beams and $c_P=0cm$ for the plate are assumed. In the second test (Fig. 4) the stretching-bending problem will be considered. For this problem, the adopted reference surface is coincident to the plate middle surface, therefore $c_B=7.5cm$ and $c_P=0cm$.

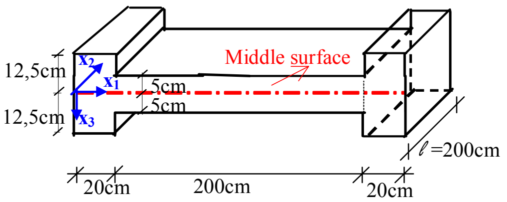

Figure 3 - Stiffened Plate – Simple Bending

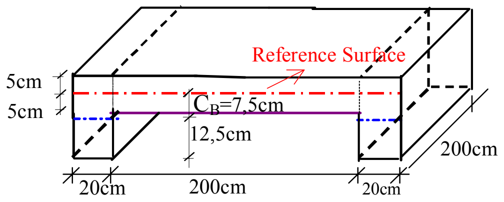

Figure 4 – Stiffened Plate – Stretching-Bending Problem

For both examples the two sides containing the beams are free, while the other two are simply supported. To run this problem, Young's modulus $E=27000kN/cm^2$, Poisson's ratio $\nu=0.0$, the yield stress σ_y=24kN/cm^2 and a tolerance of 0.1% for the convergence were adopted. The plate sides without beams as well as the beam axes (skeleton lines) have been discretized into *12* quadratic elements (Fig. 5), giving the total amount of 100 nodes. In the domain plate, 32 triangular cells were used. Rectangular cells were adopted for the approximation along the beams, for which nodes were coincident with the elements nodes. Then, each one of these rectangular cells was divided into four triangular cells as shown in Fig. (6).

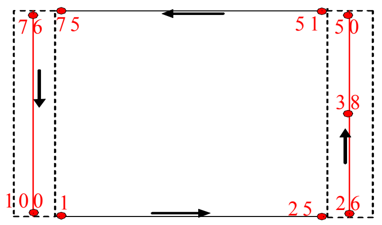

Figure 5 – Boundary Discretization

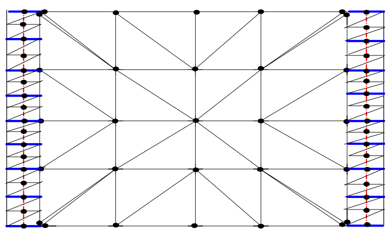
Figure 6 – Domain Discretization

In the first test the hardening modulus $K=2700kN/cm^2$ was assumed. The boundary values were conveniently prescribed to enforce constant curvatures over the entire structural element. Along the simply supported sides it was applied M_n =*360 kNm/m* along the plate boundary while M_n= *5625kNm/m* was prescribed along the beam width.

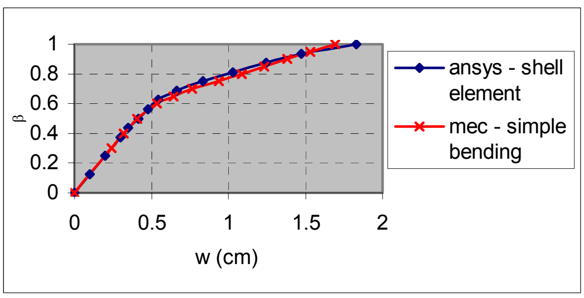

Figure 7 – Deflection at point 38

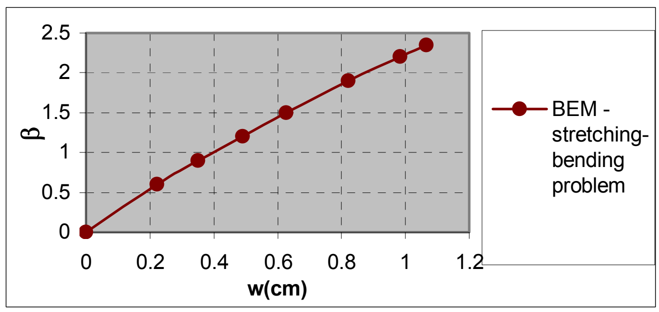

Figure 8 – Deflection at point 38

The load was applied in 12 increments. Before yielding the computed values were exactly the theoretical ones, which are giving by: $w,_2 = (M / EI)(0.5l - x_2)$ and $w = (Mx_2 / 2EI)(l - x_2)$. In order to compare the results during the non-linear process, this test was also analyzed using the commercial pack ANSYS, for which the shell elements were adopted. Figure (7) shows the deflection at the beam middle plate (point 38) during the incremental process.

In the second test (Fig 4) we have also prescribed appropriate values of moments and tractions along the simply supported sides in order to obtain constant curvature over the entire stiffened plate. Along the plate boundary was prescribed $M_n = 166.67$ *kNm/m* and along the beam width was applied $p_n = 375kN/cm$ and $M_n = 5416.67$ *kNm/m*. The adopted hardening modulus was $K = 13000kN/cm^2$. The load was applied in 15 increments to carry out the non-linear analysis. Before yielding the numerical results can be confirmed analytically. For the plate, those expressions are the same ones giving for the previous test and for the beams we must have: $w,_2 = \beta(p_n c_B - M_n)(x_2 - 0.5l)/ EI$; $u_2 = \beta p_n (x_2 - 0.5l)(c_B^2 / I + 1 / A)/ E - \beta M_n c_B (x_2 - l)/ EI$; $w = \beta(M_n - p_n c_B) x_2 (l - x_2)/(2EI)$. Along all the incremental process the internal forces computed over the beams middle surface must be equal to $M_2 = \beta(M_n - c_B p_n)$ and $N_2 = \beta p_n$. The theoretical values of both displacements and internal forces have been successfully achieved throughout the analysis. Figure 8 shows the computed deflection at point 38 along the incremental process.

It is important to point out that the convergence for displacements and other values has been confirmed by running these examples using finer meshes, which have reproduced practically the same results.

7. Conclusions

The BEM formulation for linear analysis of the zoned plate-bending problem taking into account the membrane effects has been extended in order to perform the non-linear analysis of plate reinforced by beams. In the proposed formulation equilibrium and compatibility conditions are automatically guaranteed by the global integral equations, treating this composed structure as a single body. The unknowns are obtained from a system of equations in which the bending and the plane stress problems are coupled and cannot be treated separately. The non-linear formulation, whose domain integrals are performed by discretizing the domain into cells, is obtained by incorporating initials efforts fields into the linear formulation. In the incremental-iterative procedure is used the consistent tangent operator, while Von Mises criterion is adopted to govern the non-linear behaviour. The performance of the proposed formulation has been confirmed by comparing the results with analytical solutions and also numerical solution obtained by using a well-known finite element code.

8. Acknowledgements

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